Real Variables Fall 2011 (Young) HW 3 Due Oct 3

1. Let $A \subset [0,1]$ be a positive measure set. Prove that there exist $x, y \in A$, $x \neq y$, such that $x - y \in \mathbb{Q}$.

2. Let S be a dense subset of $\mathbb{R}$. Prove that f is measurable if and only if $\{f > \alpha\}$ is measurable for all $\alpha \in S$.

3. Let $f : D \to \mathbb{R}$ be measurable with $f(x) \neq 0$ for all $x \in D$. Prove that $\frac{1}{f}$ is measurable.

4. Consider $f : D \to \mathbb{R}$ where D is a measurable set. Prove that f is measurable if
   (a) it is continuous,
   (b) it is monotone, i.e. either (i) $x \leq y$ implies $f(x) \leq f(y)$ for all $x, y \in D$, or
      (ii) $x \leq y$ implies $f(x) \geq f(y)$ for all $x, y \in D$.

5. Prove that if f is measurable, then $f^{-1}(B)$ is measurable for all Borel sets B. *Hint:* Let $\mathcal{C} = \{E \subset \mathbb{R} : f^{-1}(E) \text{ is measurable}\}$. Prove that $\mathcal{C}$ is a σ-algebra, and that it contains all Borel sets.

6. Let $f : [a,b] \to \mathbb{R}$ be a simple function. Prove that for every $\varepsilon > 0$, there is a step function h defined on $[a,b]$ such that $m\{|f - h| > \varepsilon\} < \varepsilon$.

7. Let $f : [a,b] \to \mathbb{R}$ be a bounded function (that is not necessarily measurable), and let $\mathcal{D}$ be the set of points at which f is not continuous.

(a) Prove that $\mathcal{D}$ is an F_σ. *Hint:* consider the *oscillation* at x defined to be

$$O_f(x) \;=\; \lim_{\varepsilon \to 0} \left(\sup_{y:|y-x|<\varepsilon} f(y) - \inf_{y:|y-x|<\varepsilon} f(y) \right) .$$

(b) Prove that if $m(\mathcal{D}) = 0$, then f is measurable.