Real Variables Fall 2011 (Young) HW 2 Due Sept 26

1. Let $\mathcal{A}$ be the set of all finite unions of sets of the form

$$[a,b),\ [a,\infty),\ (-\infty,b),\ \text{or } (-\infty,\infty)$$

for $a,b\in\mathbb{R},\ a<b$ together with the empty set. Prove that $\mathcal{A}$ is an *algebra*, i.e.,
(i) $\emptyset,\mathbb{R}\in\mathcal{A}$,
(ii) $A\in\mathcal{A}$ implies $A^c\in\mathcal{A}$, and
(iii) $A_1,\cdots,A_n\in\mathcal{A}$ implies $\cup_{i=1}^n A_i\in\mathcal{A}$
but that it is not a σ-algebra.

2. Give an example of a sequence of measureable sets $E_1\supset E_2\supset\dots$ with

$$m\left(\cap_{n=1}^\infty E_n\right)\neq\lim_{n\to\infty} m(E_n)\ .$$

3. Let $E\subset\mathbb{R}$ be an arbitrary measurable set with $m(E)<\infty$. Prove that for every $\varepsilon>0$, there exists a *finite* collection of intervals $\{I_1,\dots,I_n\}$ such that

$$m\left(E\ \Delta\ \cup_{i=1}^n I_i\right)<\varepsilon\ .$$

Here $\Delta=$ symmetric difference.

4. Let $A_n, n=1,2,\dots,$ be a sequence of measurable sets. A point $x\in\mathbb{R}$ is said to be in A_n *infinitely often* (abbrev. *i.o.*) if there is an infinite sequence of integers $n_1<n_2<\dots$ such that $x\in A_{n_k}$ for every k.

(a) Let $E=\{x\in\mathbb{R}: x\in A_n\ i.o.\}$. Prove that E is a measurable set.

(b) Prove that $m(E)=0$ if $\sum_{n=1}^\infty m(A_n)<\infty$.

5. (a) Compute the Lebesgue measure of the middle-third Cantor set.

Given any sequence of numbers $\{a_n\}_{n=1,2,\dots}$ with $0<a_n<1$, a (generalized) Cantor set $E=E(\{a_n\})$ can be constructed in a manner similar to the middle-third Cantor set except that instead of removing a fraction of $\frac{1}{3}$ at each stage, one removes a fraction of a_n at the nth step.

(b) Given $\{a_n\}$, what is $m(E(\{a_n\})$?

(c) Prove that for every $\alpha\in[0,1)$, there is a closed subset $E\subset[0,1]$ such that (i) E contains no intervals, and (ii) $m(E)=\alpha$.

6. (a) Let E be the set of points $x\in[0,1)$ whose decimal expansion $x=0.x_1x_2\dots$ contains somewhere the block "55555" (5 consecutive 5's). Find the measure of E.

(b) What if we require that the block "55555" appears infinitely many times?