

Real Variables Fall 2011 (Young) HW 12 Due Tues, Dec 13, 5pm
in my mailbox

1. Let f be a bounded measurable and g an integrable function on \mathbb{R}^n . Assuming that both f and g are compactly supported, prove that

- (a) $(f * g)(x)$ is defined at every x , and
- (b) $\text{supp}(f * g) \subset \text{supp}(f) + \text{supp}(g)$.

2. Let $f \in L^1(\mathbb{R}^n, \lambda)$ where λ is Lebesgue measure, and let k be a Borel function with $k > 0$ a.e. and $\int_{\infty}^{\infty} k = 1$. Assuming $f \not\equiv 0$ and $1 < p \leq \infty$, prove that

$$\|k * f\|_p < \|f\|_p .$$

3. Let (X, \mathcal{B}, μ) be a finite measure space, and let g be an integrable function such that for some constant M ,

$$\left| \int g\varphi d\mu \right| \leq M\|\varphi\|_1$$

for all simple functions φ . Prove that $g \in L^\infty$. What is an upper estimate for $\|g\|_\infty$?

4. Let (X, \mathcal{B}, μ) be a finite measure space. Prove that

$$\|f\|_p \leq \|f\|_q \cdot \mu(X)^{\frac{1}{p} - \frac{1}{q}} \quad \text{for } 1 \leq p < q \leq \infty ,$$

and conclude that $L^q \subset L^p$ for $p < q$. [Hint: for $q < \infty$, use the Hölder Inequality.]

5. Let X and Y be normed linear spaces, and let $T : X \rightarrow Y$ be a linear map. Prove that the following three properties are equivalent:

- (i) T is bounded;
- (ii) T is continuous;
- (iii) T is continuous at one point in X .

6. Let (X, \mathcal{B}, μ) be such that $X = \{1, 2, 3, \dots\}$, $\mathcal{B} = 2^X$, and μ is the counting measure. Define $\ell^1 = L^1(X, \mu)$, $\ell^\infty = L^\infty(X, \mu)$, and

$$c_0 = \{(\xi_i) \in \ell^\infty : \xi_i \rightarrow 0 \text{ as } i \rightarrow \infty\} ,$$

For each $y = (\eta_i) \in \ell^1$, define $f_y : c_0 \rightarrow \mathbb{R}$ by $f_y(\xi_i) = \sum_i \eta_i \xi_i$.

- (a) Prove that $f_y \in c_0^*$ with $\|f_y\| = \|y\|_1$.
- (b) Prove that every $f \in c_0^*$ can be obtained this way, i.e. $c_0^* \equiv \ell^1$.
- (c) Prove that (b) is false if c_0^* is replaced by $(\ell^\infty)^*$, i.e. $(\ell^\infty)^* \not\equiv \ell^1$.