**Real Variables Fall 2011 (Young) HW 11 Due Dec 5**

1. Let $X = Y = \{1, 2, 3, \dots\}$ and let $\mu$ and $\nu$ be the counting measure on $X$ and $Y$ respectively, i.e. $\mu(E) =$ cardinality of $E$. Consider the function $f : X \times Y \to \mathbb{R}$ defined by

$$f(m,n) = 1 \text{ if } m = n, \quad -1 \text{ if } m = n+1, \quad \text{and } 0 \text{ elsewhere.}$$

Compute

$$\int_Y \left[ \int_X f d\mu \right] d\nu \qquad \text{and} \qquad \int_X \left[ \int_Y f d\nu \right] d\mu \ .$$

Why doesn't Fubini's Theorem apply?

2. Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be $\sigma$-finite measure spaces, and let $f$ and $g$ be measurable functions on $X$ and $Y$ respectively. Define $h(x,y) = f(x)g(y)$.

(a) Prove that $h$ is measurable.

(b) If $f \in L^1(\mu)$ and $g \in L^1(\nu)$, prove that $h \in L^1(\mu \times \nu)$ and

$$\int h \ d(\mu \times \nu) = \int f d\mu \cdot \int g d\nu \ .$$

3. Let $f$ be Lebesgue integrable on $[0, a]$, and define

$$g(x) = \int_x^a \frac{1}{t} f(t) dt \ .$$

Prove that (i) $g$ is integrable on $[0, a]$, and (ii) $\int_0^a f(x)dx = \int_0^a g(x)dx$.