Real Variables Fall 2011 (Young) HW 10 Due Nov 28 (not Nov 21)

1. Let $(X, \mathcal{B}, \mu)$ be a general measure space, and let $f \geq 0$ be a measurable function.

(a) Prove that there is an increasing sequence of simple functions φ_n which converges pointwise to f.

(b) Prove that if $(X, \mathcal{B}, \mu)$ is σ-finite, then the sequence in (a) can be chosen so that $\mu\{\varphi_n \neq 0\} < \infty$ for each n.

(c) Without assuming σ-finiteness, prove that if f is integrable, then for every $\varepsilon > 0$, there exists $A \in \mathcal{B}$ with $\mu(A) < \infty$ such that $\int_A f > \int_X f - \varepsilon$.

2. (a) Prove Egoroff's Theorem for abstract measure spaces.

(b) Let $(X, \mathcal{B}, \mu)$ be σ-finite, and let $f_n \to f$ a.e. Prove that there exist measurable sets $A_1, A_2, \ldots$ with $\mu(X \setminus \cup_1^\infty A_k) = 0$ such that on each A_k, $f_n \to f$ uniformly.

3. Prove the following result for general measure spaces: Let $\{f_n\}$ be such that $|f_n| \leq g$ for some $g \in L^1$. Prove that if $f_n \to f$ in measure, then $f_n \to f$ in L^1.

4. (a) Show that any monotone increasing function on $\mathbb{R}$ is equal, modulo an at most countable set of points, to one that is continuous from the right.

(b) Extend the idea of cumulative distributive functions (cdf) to show that each bounded function F of bounded variation gives rise to a signed measure ν on $(\mathbb{R}, \mathcal{B})$.

(c) How is the Jordan decomposition of ν related to the positive and negative variations of F?

5. (a) Show that a measure μ on $(\mathbb{R}, \mathcal{B})$ is absolutely continuous with respect to Lebesgue measure if and only if its cdf is absolutely continuous.

(b) Prove that if μ is absolutely continuous with respect to Lebesgue measure, then its Radon-Nikodym derivative is the derivative of its cdf.

6. Let ν, μ and λ be σ-finite measures on $(X, \mathcal{B})$.

(a) Suppose $\nu \ll \mu$. Prove that if $g \in L^1(\nu)$, then

  (i) $g\frac{d\nu}{d\mu} \in L^1(\mu)$ and

  (ii) $\int g d\nu = \int (g\frac{d\nu}{d\mu}) d\mu$. [Hint: Consider simple functions.]

(b) Suppose $\nu \ll \mu$ and $\mu \ll \lambda$. Prove that $\nu \ll \lambda$, and $\frac{d\nu}{d\lambda} = \frac{d\nu}{d\mu}\frac{d\mu}{d\lambda}$.

7. Let ν and μ be finite measures on $(X, \mathcal{B})$ with $\nu \ll \mu$, and let $\lambda = \nu + \mu$. Prove that $\nu \ll \lambda$, and find the relation between $\frac{d\nu}{d\mu}$ and $\frac{d\nu}{d\lambda}$.