Real Variables Fall 2011 (Young) HW 1 Due Sept 19

Let $m^*(\cdot)$ denote the outer measure of a set.

1. Prove that $m^*(A) = 0$ if A consists of a single point.

2. Prove that $m^*(A) = 0$ if A is at most countable.

3. Given $A \subset \mathbb{R}$ and $\varepsilon > 0$, prove that there is an open set $O \supset A$ with

$$m^*(O) \leq m^*(A) + \varepsilon \ .$$

4. Give an example of a set $A \subset \mathbb{R}$ such that

$$m^*(A) \ < \ \text{Inf} \left\{ \sum_{I \in \mathcal{I}} \ell(I) \ : \ \mathcal{I} = \text{ cover of } A \text{ by a } \textit{finite} \text{ number of intervals} \right\} \ .$$

5. Prove that $m^*(\cdot)$ is translation invariant, i.e., for $A \subset \mathbb{R}$ and $y \in \mathbb{R}$, if we define

$$A + y = \{x + y : x \in A\} \ ,$$

then $m^*(A + y) = m^*(A)$ for all A and y.

6. Prove that every open set $O \subset \mathbb{R}$ is the union of at most countably many disjoint open intervals.

7. Let $\mathcal{N}$ be a σ-algebra of subsets of $\mathbb{R}$, and let $E_1, E_2, \dots \in \mathcal{N}$.
(a) Prove that there exist $F_1, F_2, \dots \in \mathcal{N}$ with $F_i \cap F_j = \emptyset$ for all $i \neq j$ such that

$$\cup_{i=1}^n E_i = \cup_{i=1}^n F_i \qquad \text{for all } n \geq 1 \ .$$

(b) Prove that there exist $G_1, G_2, \dots \in \mathcal{N}$ with $G_1 \subset G_2 \subset \dots$ such that

$$G_n = \cup_{i=1}^n E_i \qquad \text{for all } n \geq 1 \ .$$