1. If \( T \in \Lambda^k(V) \), \( \vec{v}_1, \ldots, \vec{v}_k \) is a set of \( k \) linearly dependent vectors on \( V \), prove \( T(\vec{v}_1, \ldots, \vec{v}_k) = 0 \)

**Solution:** Since \( \vec{v}_1, \ldots, \vec{v}_k \) is a set of \( k \) linearly dependent vectors, there exists \( a_1, \ldots, a_k \in \mathbb{F} \) such that
\[
a_1 \vec{v}_1 + \ldots + a_k \vec{v}_k = \vec{0}
\]
and at least one \( a_i \neq 0 \). Then it implies we can write
\[
\vec{v}_i = -\frac{a_1}{a_i} \vec{v}_1 - \ldots - \frac{a_{i-1}}{a_i} \vec{v}_{i-1} - \frac{a_i+1}{a_i} \vec{v}_{i+1} - \ldots - \frac{a_k}{a_i} \vec{v}_k
\]
So
\[
T(\vec{v}_1, \ldots, \vec{v}_n)
= T(\vec{v}_1, \ldots, \vec{v}_i, \ldots, \vec{v}_k)
= a_1 \frac{T(\vec{v}_1, \ldots, \vec{v}_{i-1}, \vec{v}_1, \vec{v}_{i+1}, \ldots, \vec{v}_k) - \cdots - a_i \frac{T(\vec{v}_1, \ldots, \vec{v}_{i-1}, \vec{v}_{i-1}, \vec{v}_{i+1}, \ldots, \vec{v}_k) - \cdots - a_k \frac{T(\vec{v}_1, \ldots, \vec{v}_{i-1}, \vec{v}_k, \vec{v}_{i+1}, \ldots, \vec{v}_k) - \cdots - 0 = 0
\]

2. If \( T \in \Lambda^k(V) \) and \( S \in \Lambda^l(V) \), prove \( T \wedge S = (-1)^{kl} S \wedge T \)

**Solution:** If \( \{\vec{v}_1, \ldots, \vec{v}_n\} \) is a basis for \( V \) and \( \phi_1, \ldots, \phi_n \) is the dual basis for \( V^* \). First we show that \( \phi_i \wedge \phi_j = -\phi_j \wedge \phi_i \):
\[
\phi_i \wedge \phi_j(\vec{u}_1, \vec{u}_2)
= \sum_{\sigma \in S_2} \text{sgn}(\sigma) \phi_i(\vec{u}_{\sigma(1)}) \phi_j(\vec{u}_{\sigma(2)})
= \phi_i(\vec{u}_1) \phi_j(\vec{u}_2) - \phi_i(\vec{u}_2) \phi_j(\vec{u}_1)
= - \phi_j(\vec{u}_1) \phi_i(\vec{u}_2) + \phi_j(\vec{u}_2) \phi_i(\vec{u}_1)
= - \sum_{\sigma \in S_2} \text{sgn}(\sigma) \phi_j(\vec{u}_{\sigma(1)}) \phi_i(\vec{u}_{\sigma(2)})
= - \phi_j \wedge \phi_i(\vec{u}_1, \vec{u}_2)
\]
This implies
\[ \phi_{i_1} \wedge ... \wedge \phi_{i_k} \wedge \phi_j = (-1)^k \phi_j \wedge ... \wedge \phi_j \wedge \phi_{i_1} \wedge ... \wedge \phi_{i_k} \]

If \( T = \sum_{1 \leq i_1 < ... < i_k \leq n} T_{i_1...i_k} \phi_{i_1} \wedge ... \wedge \phi_{i_k} \) and \( S = \sum_{1 \leq j_1 < ... < j_l \leq n} S_{j_1...j_l} \phi_{j_1} \wedge ... \wedge \phi_{j_l} \), we get

\[ T \wedge S = \left( \sum_{1 \leq i_1 < ... < i_k \leq n} T_{i_1...i_k} \phi_{i_1} \wedge ... \wedge \phi_{i_k} \right) \wedge \left( \sum_{1 \leq j_1 < ... < j_l \leq n} S_{j_1...j_l} \phi_{j_1} \wedge ... \wedge \phi_{j_l} \right) \]

\[ = (-1)^k \sum_{1 \leq i_1 < ... < i_k \leq n} \sum_{1 \leq j_1 < ... < j_l \leq n} T_{i_1...i_k} S_{j_1...j_l} \phi_{j_1} \wedge ... \wedge \phi_{j_l} \wedge \phi_{i_1} \wedge ... \wedge \phi_{i_k} \]

\[ = (-1)^k \sum_{1 \leq j_1 < ... < j_l \leq n} \sum_{1 \leq i_1 < ... < i_k \leq n} S_{j_1...j_l} T_{i_1...i_k} \phi_{j_1} \wedge ... \wedge \phi_{j_l} \wedge \phi_{i_1} \wedge ... \wedge \phi_{i_k} \]

\[ = (-1)^k \left( \sum_{1 \leq j_1 < ... < j_l \leq n} S_{j_1...j_l} \phi_{j_1} \wedge ... \wedge \phi_{j_l} \right) \wedge \left( \sum_{1 \leq i_1 < ... < i_k \leq n} T_{i_1...i_k} \phi_{i_1} \wedge ... \wedge \phi_{i_k} \right) \]

\[ = (-1)^k S \wedge T \]

3. If \( \{ \vec{v}_1, ..., \vec{v}_n \} \) is a basis for a \( n \)-dimensional vector space \( V \) and \( \{ \phi_1, ..., \phi_n \} \) is the dual basis for \( V^* \), prove for each \( \sigma \in S_n \),

\[ \phi_{\sigma(1)} \wedge ... \wedge \phi_{\sigma(n)} = sgn(\sigma) \phi_1 \wedge ... \wedge \phi_n \]

**Solution:**

Method I: By (2), \( \phi_i \wedge \phi_j = -\phi_j \wedge \phi_i \). We can get from \( \phi_{\sigma(1)} \wedge ... \wedge \phi_{\sigma(n)} \) to \( \phi_1 \wedge ... \wedge \phi_n \) by odd number of transpositions if \( sgn(\sigma) = -1 \), and even number of transpositions if \( sgn(\sigma) = +1 \), so we get the conclusion.
Method II:

\[ \phi_{\sigma(1)} \wedge ... \wedge \phi_{\sigma(n)}(\vec{v}_1, ..., \vec{v}_n) \]

\[ = n! \text{Alt}(\phi_{\sigma(1)} \otimes ... \otimes \phi_{\sigma(n)})(\vec{v}_1, ..., \vec{v}_n) \]

\[ = \sum_{\tau \in S_n} \text{sgn}(\tau) \phi_{\sigma(1)}(\vec{v}_{\tau(1)}) ... \phi_{\sigma(n)}(\vec{v}_{\tau(n)}) \]

\[ = \sum_{\tau \in S_n} \text{sgn}(\tau) \phi_{\sigma(1)}(\vec{v}_{\tau(1)}) ... \phi_{\sigma(n)}(\vec{v}_{\tau(n)}) \]

\[ = \text{sgn}(\sigma) \sum_{\tau \in S_n} \text{sgn}(\tau) \phi_{\sigma(1)}(\vec{v}_{\tau(1)}) ... \phi_{\sigma(n)}(\vec{v}_{\tau(n)}) \]

\[ = \text{sgn}(\sigma) \sum_{\tau \in S_n} \text{sgn}(\tau) \phi_{\sigma(1)}(\vec{v}_{\tau(1)}) ... \phi_{\sigma(n)}(\vec{v}_{\tau(n)}) \]

\[ = \text{sgn}(\sigma) \sum_{\tau \in S_n} \phi_{\tau(1)}(\vec{v}_{\tau(1)}) ... \phi_{\tau(n)}(\vec{v}_{\tau(n)}) \]

\[ = \text{sgn}(\sigma) \text{Alt}(\phi_1 \otimes ... \otimes \phi_n)(\vec{v}_1, ..., \vec{v}_n) \]

\[ = \text{sgn}(\sigma) \phi_1 \wedge ... \wedge \phi_n(\vec{v}_1, ..., \vec{v}_n) \]

4. \( T \in T^k(V) \) is called a symmetric \( k \)-tensor if for any \( \vec{v}_1, ..., \vec{v}_k \in V \) and any \( \sigma \in S_k \), \( T(\vec{v}_{\sigma(1)}, ..., \vec{v}_{\sigma(k)}) = T(\vec{v}_1, ..., \vec{v}_k) \). The set of all symmetric \( k \)-tensors form the subspace \( \text{Sym}^k(V) \) of \( T^k(V) \).

(i). Given any \( T \in T^k(V) \), define \( \text{Sym}(T) \) to be the \( k \)-tensor:

\[ \text{Sym}(T)(\vec{v}_1, ..., \vec{v}_k) = \frac{1}{k!} \sum_{\sigma \in S_k} T(\vec{v}_{\sigma(1)}, ..., \vec{v}_{\sigma(k)}) \]

Prove \( \text{Sym}(T) \in \text{Sym}^k(V) \).

Solution:

For any \( \tau \in S_k \),

\[ \text{Sym}(T)(\vec{v}_{\tau(1)}, ..., \vec{v}_{\tau(k)}) \]

\[ = \frac{1}{k!} \sum_{\sigma \in S_k} T(\vec{v}_{\sigma(1)}, ..., \vec{v}_{\sigma(1)}) \]

\[ = \frac{1}{k!} \sum_{\sigma \in S_k} T(\vec{v}_{\sigma(1)}, ..., \vec{v}_{\sigma(1)}) \]

\[ = \text{Sym}(T)(\vec{v}_1, ..., \vec{v}_k) \]
The second last equality is because for fixed $\tau \in S_k$, there is a bijection

$$S_k \rightarrow S_k$$

given by $\sigma \mapsto \sigma \tau$

(ii). If $T \in Sym^k(V)$, prove $Sym(T) = T$

**Solution:**
If $T \in Sym^k(V),$

$$Sym(T)(\vec{v}_1, ..., \vec{v}_k)$$

$$= \frac{1}{k!} \sum_{\sigma \in S_k} T(\vec{v}_{\sigma(1)}, ..., \vec{v}_{\sigma(k)})$$

$$= \frac{1}{k!} \sum_{\sigma \in S_k} T(\vec{v}_1, ..., \vec{v}_k)$$

$$= \frac{1}{k!} \times k! T(\vec{v}_1, ..., \vec{v}_k)$$

$$= T(\vec{v}_1, ..., \vec{v}_k)$$

(iii). $T \in T^2(V)$, prove $T = Sym(T) + Alt(T)$

**Solution:** If $T \in T^2(V),$

$$Sym(T)(\vec{v}_1, \vec{v}_2) = \frac{1}{2!} \sum_{\sigma \in S_2} T(\vec{v}_{\sigma(1)}, \vec{v}_{\sigma(2)}) = \frac{T(\vec{v}_1, \vec{v}_2) + T(\vec{v}_2, \vec{v}_1)}{2}$$

$$Alt(T)(\vec{v}_1, \vec{v}_2) = \frac{1}{2!} \sum_{\sigma \in S_2} sgn(\sigma)T(\vec{v}_{\sigma(1)}, \vec{v}_{\sigma(2)}) = \frac{T(\vec{v}_1, \vec{v}_2) - T(\vec{v}_2, \vec{v}_1)}{2}$$

So

$$[Sym(T) + Alt(T)](\vec{v}_1, \vec{v}_2)$$

$$= Sym(T)(\vec{v}_1, \vec{v}_2) + Alt(T)(\vec{v}_1, \vec{v}_2)$$

$$= \frac{T(\vec{v}_1, \vec{v}_2) + T(\vec{v}_2, \vec{v}_1)}{2} + \frac{T(\vec{v}_1, \vec{v}_2) - T(\vec{v}_2, \vec{v}_1)}{2}$$

$$= T(\vec{v}_1, \vec{v}_2)$$

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We conclude \( T = Sym(T) + Alt(T) \)

(iv). Prove \( Sym^2(V) \cap \Lambda^2(V) = \{0\} \)

**Solution:** If \( T \in Sym^2(V) \), then \( T(\vec{v}_1, \vec{v}_2) = T(\vec{v}_2, \vec{v}_1) \). If \( T \in \Lambda^2(V) \), then \( T(\vec{v}_1, \vec{v}_2) = -T(\vec{v}_2, \vec{v}_1) \).

Combining these two equations, we see if \( T \in Sym^2(V) \cap \Lambda^2(V) \), then \( T(\vec{v}_1, \vec{v}_2) = 0 \) for any \( \vec{v}_1, \vec{v}_2 \in V \).

(v). \( T \in T^2(V) \). Prove \( T \) can be decomposed as the sum of a symmetric 2-tensor and an alternating 2-tensor in a unique way. (Remark: This is equivalent to say \( T^2(V) = Sym^2(V) \oplus \Lambda^2(V) \))

**Solution:** By (iii), we know \( T = Sym(T) + Alt(T) \), which gives one such decomposition. We next prove this decomposition is unique:

If \( T = \omega_1 + \eta_1 = \omega_2 + \eta_2 \), where \( \omega_1, \omega_2 \in Sym^2(V) \) and \( \eta_1, \eta_2 \in \Lambda^2(V) \), then

\[
\omega_1 - \omega_2 = \eta_2 - \eta_1
\]

The left side belongs to \( Sym^2(V) \) while the right side belongs to \( \Lambda^2(V) \), so

\[
\omega_1 - \omega_2 = \eta_2 - \eta_1 \in Sym^2(V) \cap \Lambda^2(V) = \{0\}
\]

We conclude \( \omega_1 = \omega_2 \) and \( \eta_1 = \eta_2 \), so the decomposition is unique.

(vi). \( \{\vec{u}_1, ..., \vec{u}_n\} \) is a basis of \( V \). \( T \in T^2(V) \). \( A = (a_{ij}) \) is the \( n \times n \) matrix such that \( a_{ij} = T(\vec{u}_i, \vec{u}_j) \). A square matrix \( A \) is called skew-symmetric if \( A^t = -A \), where \( A^t \) denotes the transpose of \( A \). Prove \( T \in Sym^2(V) \) if and only if \( A \) is a symmetric matrix, and \( T \in \Lambda^2(V) \) if and only if \( A \) is a skew-symmetric matrix.

**Solution:**

\( T \in Sym^2(V) \iff T(\vec{e}_i, \vec{e}_j) = T(\vec{e}_j, \vec{e}_i) \iff a_{ij} = a_{ji} \iff A \) is symmetric.

\( T \in \Lambda^2(V) \iff T(\vec{e}_i, \vec{e}_j) = -T(\vec{e}_j, \vec{e}_i) \iff a_{ij} = -a_{ji} \iff A \) is skew-symmetric.

(vii). Prove Each \( n \times n \) matrix can be written as a sum of a symmetric matrix and a skew-symmetric matrix in a unique way.

**Solution:** Fixing a basis \( \{\vec{e}_1, ..., \vec{e}_n\} \), there is a one-to-one correspondence between 2-tensors on \( V \) and \( n \times n \) matrices as described in (vi). So given \( n \times n \) matrix \( A \), it corresponds to some 2-tensor \( T \). In (v) we proved \( T = \omega + \eta \) can be decomposed as a sum of a symmetric 2-tensor \( \omega \) and an alternating 2-tensor.
\( \eta \) in a unique way, and in (vi) we proved \( \omega \in Sym^2(V) \) if and only if its matrix \( B \) is a symmetric matrix, and \( \eta \in \Lambda^2(V) \) if and only if its matrix \( C \) is a skew-symmetric matrix, so \( A = B + C \) can be written as a sum of a symmetric matrix and a skew-symmetric matrix in a unique way.

(viii). Write the matrix \[
\begin{bmatrix}
1 & 2 & 3 \\
4 & 5 & 6 \\
7 & 8 & 9
\end{bmatrix}
\]
as the sum of a symmetric matrix and a skew-symmetric matrix.

Solution: If the matrix of \( T \) is \( A = (T(\vec{e}_i, \vec{e}_j)) \), then the matrix for \( Sym(T) \) is \( \left( \frac{T(\vec{e}_i, \vec{e}_j) + T(\vec{e}_j, \vec{e}_i)}{2} \right) = \frac{1}{2}(A + A^t) \) and the matrix for \( Alt(T) \) is \( \left( \frac{T(\vec{e}_i, \vec{e}_j) - T(\vec{e}_j, \vec{e}_i)}{2} \right) = \frac{1}{2}(A - A^t) \).

So \( A = \frac{1}{2}(A + A^t) + \frac{1}{2}(A - A^t) \). Applying this formula we get
\[
\begin{bmatrix}
1 & 2 & 3 \\
4 & 5 & 6 \\
7 & 8 & 9
\end{bmatrix} = \begin{bmatrix}
1 & 3 & 5 \\
3 & 5 & 7 \\
5 & 7 & 9
\end{bmatrix} + \begin{bmatrix}
0 & -1 & -2 \\
1 & 0 & -1 \\
2 & 1 & 0
\end{bmatrix}
\]