1. If $T \in \Lambda^k(V)$, $\vec{v}_1, ..., \vec{v}_k$ is a set of k linearly dependent vectors on V, prove $T(\vec{v}_1, ..., \vec{v}_k) = 0$

   Solution: Since $\vec{v}_1, ..., \vec{v}_k$ is a set of k linearly dependent vectors, there exists $a_1, ..., a_k \in \mathbb{F}$ such that

$$a_1\vec{v}_1 + ... + a_k\vec{v}_k = \vec{0}$$

   and at least one $a_i \neq 0$. Then it implies we can write

$$\vec{v}_i = -\frac{a_1}{a_i}\vec{v}_1 - ... - \frac{a_{i-1}}{a_i}\vec{v}_{i-1} - \frac{a_{i+1}}{a_i}\vec{v}_{i+1} - ... - \frac{a_k}{a_i}\vec{v}_k$$

   So

$$\begin{aligned}
&T(\vec{v}_1, ..., \vec{v}_n)\\
=&T(\vec{v}_1, ..., \vec{v}_i, ..., \vec{v}_k)\\
=&T(\vec{v}_1, ..., -\frac{a_1}{a_i}\vec{v}_1 - ... - \frac{a_{i-1}}{a_i}\vec{v}_{i-1} - \frac{a_{i+1}}{a_i}\vec{v}_{i+1} - ... - \frac{a_k}{a_i}\vec{v}_k, ..., \vec{v}_k)\\
=&-\frac{a_1}{a_i}T(\vec{v}_1, ...\vec{v}_{i-1}, \vec{v}_1, \vec{v}_{i+1}, ..., \vec{v}_k) - ... - \frac{a_{i-1}}{a_i}T(\vec{v}_1, ...\vec{v}_{i-1}, \vec{v}_{i-1}, \vec{v}_{i+1}, ..., \vec{v}_k)\\
&-\frac{a_{i+1}}{a_i}T(\vec{v}_1, ...\vec{v}_{i-1}, \vec{v}_{i+1}, \vec{v}_{i+1}, ..., \vec{v}_k) - ... - \frac{a_k}{a_i}T(\vec{v}_1, ...\vec{v}_{i-1}, \vec{v}_k, \vec{v}_{i+1}, ..., \vec{v}_k)\\
=&-0 - ... - 0 - 0 - ... - 0\\
=&0
\end{aligned}$$

2. If $T \in \Lambda^k(V)$ and $S \in \Lambda^l(V)$, prove $T \wedge S = (-1)^{kl} S \wedge T$

   Solution: If $\{\vec{v}_1, ..., \vec{v}_n\}$ is a basis for V and $\phi_1, ..., \phi_n$ is the dual basis for V^*. First we show that $\phi_i \wedge \phi_j = -\phi_j \wedge \phi_i$:

$$\begin{aligned}
&\phi_1 \wedge \phi_j(\vec{u}_1, \vec{u}_2)\\
=&\sum_{\sigma \in S_2} sng(\sigma)\phi_i(\vec{u}_{\sigma(1)})\phi_j(\vec{u}_{\sigma(2)})\\
=&\phi_i(\vec{u}_1)\phi_j(\vec{u}_2) - \phi_i(\vec{u}_2)\phi_j(\vec{u}_1)\\
=&-\phi_j(\vec{u}_1)\phi_i(\vec{u}_2) + \phi_j(\vec{u}_2)\phi_i(\vec{u}_1)\\
=&-\sum_{\sigma \in S_2} sgn(\sigma)\phi_j(\vec{u}_{\sigma(1)})\phi_i(\vec{u}_{\sigma(2)})\\
=&-\phi_j \wedge \phi_i(\vec{u}_1, \vec{u}_2)
\end{aligned}$$

This implies

$$\phi_{i_1} \wedge ... \wedge \phi_{i_k} \wedge \phi_{j_1} \wedge ... \wedge \phi_{j_l} = (-1)^{kl} \phi_{j_1} \wedge ... \wedge \phi_{j_l} \wedge \phi_{i_1} \wedge ... \wedge \phi_{i_k}$$

If $T = \sum_{1 \le i_1 < ... < i_k \le n} T_{i_1 ... i_k} \phi_{i_1} \wedge ... \wedge \phi_{i_k}$ and $S = \sum_{1 \le j_1 < ... < j_l \le n} S_{j_1 ... j_l} \phi_{j_1} \wedge ... \wedge \phi_{j_l}$, we get

$$\begin{aligned}
& T \wedge S \\
=& (\sum_{1 \le i_1 < ... < i_k \le n} T_{i_1 ... i_k} \phi_{i_1} \wedge ... \wedge \phi_{i_k}) \wedge (\sum_{1 \le j_1 < ... < j_l \le n} S_{j_1 ... j_l} \phi_{j_1} \wedge ... \wedge \phi_{j_l}) \\
=& \sum_{1 \le i_1 < ... < i_k \le n} \sum_{1 \le j_1 < ... < j_l \le n} T_{i_1 ... i_k} S_{j_1 ... j_l} \phi_{i_1} \wedge ... \wedge \phi_{i_k} \wedge \phi_{j_1} \wedge ... \wedge \phi_{j_l} \\
=& (-1)^{kl} \sum_{1 \le i_1 < ... < i_k \le n} \sum_{1 \le j_1 < ... < j_l \le n} T_{i_1 ... i_k} S_{j_1 ... j_l} \phi_{j_1} \wedge ... \wedge \phi_{j_l} \wedge \phi_{i_1} \wedge ... \wedge \phi_{i_k} \\
=& (-1)^{kl} \sum_{1 \le j_1 < ... < j_l \le n} \sum_{1 \le i_1 < ... < i_k \le n} S_{j_1 ... j_l} T_{i_1 ... i_k} \phi_{j_1} \wedge ... \wedge \phi_{j_l} \wedge \phi_{i_1} \wedge ... \wedge \phi_{i_k} \\
=& (-1)^{kl} (\sum_{1 \le j_1 < ... < j_l \le n} S_{j_1 ... j_l} \phi_{j_1} \wedge ... \wedge \phi_{j_l}) \wedge (\sum_{1 \le i_1 < ... < i_k \le n} T_{i_1 ... i_k} \phi_{i_1} \wedge ... \wedge \phi_{i_k}) \\
=& (-1)^{kl} S \wedge T
\end{aligned}$$

3. If $\{\vec{v}_1, ..., \vec{v}_n\}$ is a basis for a n-dimensional vector space V and $\{\phi_1, ..., \phi_n\}$ is the dual basis for V^*, prove for each $\sigma \in S_n$,

$$\phi_{\sigma(1)} \wedge ... \wedge \phi_{\sigma(n)} = sgn(\sigma) \phi_1 \wedge ... \wedge \phi_n$$

Solution:

Method I: By (2), $\phi_i \wedge \phi_j = -\phi_j \wedge \phi_i$. We can get from $\phi_{\sigma(1)} \wedge ... \wedge \phi_{\sigma(n)}$ to $\phi_1 \wedge ... \wedge \phi_n$ by odd number of transpositions if $sgn(\sigma) = -1$, and even number of transpositions if $sgn(\sigma) = +1$, so we get the conclusion.

Method II:

$$\begin{aligned}
&\phi_{\sigma(1)} \wedge ... \wedge \phi_{\sigma(n)}(\vec{v}_1, ..., \vec{v}_n) \\
=&n!Alt(\phi_{\sigma(1)} \otimes ... \otimes \phi_{\sigma(n)})(\vec{v}_1, ..., \vec{v}_n) \\
=&\sum_{\tau \in S_n} sgn(\tau)\phi_{\sigma(1)}(\vec{v}_{\tau(1)})...\phi_{\sigma(n)}(\vec{v}_{\tau(n)}) \\
=&\sum_{\tau \in S_n} sgn(\tau\sigma)\phi_{\sigma(1)}(\vec{v}_{\tau\sigma(1)})...\phi_{\sigma(n)}(\vec{v}_{\tau\sigma(n)}) \\
=&sgn(\sigma)\sum_{\tau \in S_n} sgn(\tau)\phi_{\sigma(1)}(\vec{v}_{\tau(\sigma(1))})...\phi_{\sigma(n)}(\vec{v}_{\tau(\sigma(n))}) \\
=&sgn(\sigma)\sum_{\tau \in S_n} sgn(\tau)\phi_1(\vec{v}_{\tau(1)})...\phi_n(\vec{v}_{\tau(n)}) \\
=&sgn(\sigma)n!Alt(\phi_1 \otimes .. \otimes \phi_n)(\vec{v}_1, ..., \vec{v}_n) \\
=&sgn(\sigma)\phi_1 \wedge ... \wedge \phi_n(\vec{v}_1, ..., \vec{v}_n)
\end{aligned}$$

4. $T \in T^k(V)$ is called a symmetric k-tensor if for any $\vec{v}_1, ..., \vec{v}_k \in V$ and any $\sigma \in S_k$, $T(\vec{v}_{\sigma(1)}, ..., \vec{v}_{\sigma(k)}) = T(\vec{v}_1, ..., \vec{v}_k)$. The set of all symmetric k-tensors form the subspace $Sym^k(V)$ of $T^k(V)$.

   (i). Given any $T \in T^k(V)$, define $Sym(T)$ to be the k-tensor:

$$Sym(T)(\vec{v}_1, ..., \vec{v}_k) = \frac{1}{k!}\sum_{\sigma \in S_k} T(\vec{v}_{\sigma(1)}, ..., \vec{v}_{\sigma(k)})$$

   Prove $Sym(T) \in Sym^k(V)$.

   Solution:

   For any $\tau \in S_k$,

$$\begin{aligned}
&Sym(T)(\vec{v}_{\tau(1)}, ..., \vec{v}_{\tau(k)}) \\
=&\frac{1}{k!}\sum_{\sigma \in S_n} T(\vec{v}_{\sigma\tau(1)}, ..., \vec{v}_{\sigma\tau(1)}) \\
=&\frac{1}{k!}\sum_{\sigma \in S_n} T(\vec{v}_{\sigma(1)}, ..., \vec{v}_{\sigma(1)}) \\
=&Sym(T)(\vec{v}_1, ..., \vec{v}_k)
\end{aligned}$$

The second last equality is because for fixed $\tau \in S_k$, there is a bijection

$$S_k \longrightarrow S_k$$

given by $\sigma \mapsto \sigma\tau$

(ii). If $T \in Sym^k(V)$, prove $Sym(T) = T$

Solution:

If $T \in Sym^k(V)$,

$$\begin{aligned}
&Sym(T)(\vec{v}_1, ..., \vec{v}_k) \\
=&\frac{1}{k!}\sum_{\sigma \in S_k} T(\vec{v}_{\sigma(1)}, ..., \vec{v}_{\sigma(k)}) \\
=&\frac{1}{k!}\sum_{\sigma \in S_k} T(\vec{v}_1, ..., \vec{v}_k) \\
=&\frac{1}{k!} \times k! T(\vec{v}_1, ..., \vec{v}_k) \\
=&T(\vec{v}_1, ..., \vec{v}_k)
\end{aligned}$$

(iii). $T \in T^2(V)$, prove $T = Sym(T) + Alt(T)$

Solution: If $T \in T^2(V)$,

$$Sym(T)(\vec{v}_1, \vec{v}_2) = \frac{1}{2!}\sum_{\sigma \in S_2} T(\vec{v}_{\sigma(1)}, \vec{v}_{\sigma(2)}) = \frac{T(\vec{v}_1, \vec{v}_2) + T(\vec{v}_2, \vec{v}_1)}{2}$$

$$Alt(T)(\vec{v}_1, \vec{v}_2) = \frac{1}{2!}\sum_{\sigma \in S_2} sgn(\sigma) T(\vec{v}_{\sigma(1)}, \vec{v}_{\sigma(2)}) = \frac{T(\vec{v}_1, \vec{v}_2) - T(\vec{v}_2, \vec{v}_1)}{2}$$

So

$$\begin{aligned}
&[Sym(T) + Alt(T)](\vec{v}_1, \vec{v}_2) \\
=&Sym(T)(\vec{v}_1, \vec{v}_2) + Alt(T)(\vec{v}_1, \vec{v}_2) \\
=&\frac{T(\vec{v}_1, \vec{v}_2) + T(\vec{v}_2, \vec{v}_1)}{2} + \frac{T(\vec{v}_1, \vec{v}_2) - T(\vec{v}_2, \vec{v}_1)}{2} \\
=&T(\vec{v}_1, \vec{v}_2)
\end{aligned}$$

We conclude $T = Sym(T) + Alt(T)$

(iv). Prove $Sym^2(V) \cap \Lambda^2(V) = \{0\}$

Solution: If $T \in Sym^2(V)$, then $T(\vec{v}_1, \vec{v}_2) = T(\vec{v}_2, \vec{v}_1)$. If $T \in \Lambda^2(V)$, then $T(\vec{v}_1, \vec{v}_2) = -T(\vec{v}_2, \vec{v}_1)$.

Combining these two equations, we see if $T \in Sym^2(V) \cap \Lambda^2(V)$, then $T(\vec{v}_1, \vec{v}_2) = 0$ for any $\vec{v}_1, \vec{v}_2 \in V$.

(v). $T \in T^2(V)$. Prove T can be decomposed as the sum of a symmetric 2-tensor and an alternating 2-tensor in a unique way. (Remark: This is equivalent to say $T^2(V) = Sym^2(V) \oplus \Lambda^2(V)$)

Solution: By (iii), we know $T = Sym(T) + Alt(T)$, which gives one such decomposition. We next prove this decomposition is unique:

If $T = \omega_1 + \eta_1 = \omega_2 + \eta_2$, where $\omega_1, \omega_2 \in Sym^2(V)$ and $\eta_1, \eta_2 \in \Lambda^2(V)$, then

$$\omega_1 - \omega_2 = \eta_2 - \eta_1$$

The left side belongs to $Sym^2(V)$ while the right side belongs to $\Lambda^2(V)$, so

$$\omega_1 - \omega_2 = \eta_2 - \eta_1 \in Sym^2(V) \cap \Lambda^2(V) = \{0\}$$

We conclude $\omega_1 = \omega_2$ and $\eta_1 = \eta_2$, so the decomposition is unique.

(vi). $\{\vec{u}_1, ..., \vec{u}_n\}$ is a basis of V. $T \in T^2(V)$. $A = (a_{ij})$ is the $n \times n$ matrix such that $a_{ij} = T(\vec{u}_i, \vec{u}_j)$. A square matrix A is called skew-symmetric if $A^t = -A$, where A^t denotes the transpose of A. Prove $T \in Sym^2(V)$ if and only if A is a symmetric matrix, and $T \in \Lambda^2(V)$ if and only if A is a skew-symmetric matrix.

Solution:

$T \in Sym^2(V) \iff T(\vec{e}_i, \vec{e}_j) = T(\vec{e}_j, \vec{e}_i) \iff a_{ij} = a_{ji} \iff A$ is symmetric.

$T \in \Lambda^2(V) \iff T(\vec{e}_i, \vec{e}_j) = -T(\vec{e}_j, \vec{e}_i) \iff a_{ij} = -a_{ji} \iff A$ is skew-symmetric.

(vii). Prove Each $n \times n$ matrix can be written as a sum of a symmetric matrix and a skew-symmetric matrix in a unique way.

Solution: Fixing a basis $\{\vec{e}_1, ..., \vec{e}_n\}$, there is a one-to-one correspondence between 2-tensors on V and $n \times n$ matrices as described in (vi). So given $n \times n$ matrix A, it corresponds to some 2-tensor T. In (v) we proved $T = \omega + \eta$ can be decomposed as a sum of a symmetric 2-tensor ω and an alternating 2-tensor

η in a unique way, and in (vi) we proved $\omega \in Sym^2(V)$ if and only if its matrix B is a symmetric matrix, and $\eta \in \Lambda^2(V)$ if and only if its matrix C is a skew-symmetric matrix, so $A = B + C$ can be written as a sum of a symmetric matrix and a skew-symmetric matrix in a unique way.

(viii). Write the matrix $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ as the sum of a symmetric matrix and a skew-symmetric matrix.

Solution: If the matrix of T is $A = (T(\vec{e}_i, \vec{e})_j)$, then the matrix for $Sym(T)$ is $(\frac{T(\vec{e}_i,\vec{e})_j + T(\vec{e}_j,\vec{e})_i}{2}) = \frac{1}{2}(A + A^t)$ and the matrix for $Alt(T)$ is $(\frac{T(\vec{e}_i,\vec{e})_j - T(\vec{e}_j,\vec{e})_i}{2}) = \frac{1}{2}(A - A^t)$.

So $A = \frac{1}{2}(A + A^t) + \frac{1}{2}(A - A^t)$. Applying this formula we get

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 5 \\ 3 & 5 & 7 \\ 5 & 7 & 9 \end{bmatrix} + \begin{bmatrix} 0 & -1 & -2 \\ 1 & 0 & -1 \\ 2 & 1 & 0 \end{bmatrix}$$