1. Given $\vec{a} = (a_1, a_2, a_3), \vec{b} = (b_1, b_2, b_3) \in \mathbb{R}^3$, define a map

$$T : \mathbb{R}^3 \longrightarrow \mathbb{R}$$
$$\vec{v} = (x, y, z) \mapsto det \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ x & y & z \end{bmatrix}$$

(i). Prove $T \in T^1(\mathbb{R}^3)$

Solution: For any $(x, y, z), (x', y', z') \in \mathbb{R}^3$, $\lambda, \mu \in \mathbb{R}$,

$$\begin{aligned} T(\lambda(x,y,z) + \mu(x',y',z')) =& T(\lambda x + \mu x', \lambda y + \mu y', \lambda z + \mu z') \\ =& \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ \lambda x + \mu x' & \lambda y + \mu y' & \lambda z + \mu z' \end{vmatrix} \\ =& \lambda \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ x & y & z \end{vmatrix} + \mu \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ x' & y' & z' \end{vmatrix} \\ =& \lambda T(x,y,z) + \mu T(x',y',z') \end{aligned}$$

(ii). In (i) you proved $T \in T^1(\mathbb{R}^3)$, so by our discussion in class, T can be represented by

$$T(\vec{x}) = \vec{z}.\vec{x}$$

for some unique $\vec{z} \in \mathbb{R}^3$. Prove $\vec{z} = \vec{a} \times \vec{b}$.

Solution:

$$\begin{aligned} z =& (T(1,0,0), T(0,1,0), T(0,0,1)) \\ =& \left(\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ 1 & 0 & 0 \end{vmatrix}, \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ 0 & 1 & 0 \end{vmatrix}, \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ 0 & 0 & 1 \end{vmatrix} \right) \\ =& (a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1) \\ =& \vec{a} \times \vec{b} \end{aligned}$$

2. We can generalize the idea in (1) to define the cross product in $\mathbb{R}^n$:

   Given $\vec{a}_1 = (a_{1,1}, ..., a_{1,n}), ..., \vec{a}_{n-1} = (a_{n-1,1}, ..., a_{n-1,n}) \in \mathbb{R}^n$, define the map

$$T : \mathbb{R}^n \longrightarrow \mathbb{R}$$

$$\vec{x} = (x_1, x_2, ..., x_n) \mapsto det \begin{bmatrix} \vec{a}_1 \\ ... \\ \vec{a}_{n-1} \\ \vec{x} \end{bmatrix} = det \begin{bmatrix} a_{1,1} & ... & a_{1,n} \\ ... & ... & ... \\ a_{n-1,1} & ... & a_{n-1,n} \\ x_1 & ... & x_n \end{bmatrix}$$

   Similar to (1), $T \in T^1(\mathbb{R}^n)$ and there is unique $z \in \mathbb{R}^n$ such that $T(\vec{x}) = \vec{z}.\vec{x}$. Define $\vec{a}_1 \times ... \times \vec{a}_{n-1} = \vec{z}$.

   Compute $(1, 2, 3, 4) \times (2, 3, 3, 1) \times (0, 2, 4, 6) \in \mathbb{R}^4$.

   Solution:

$$\begin{aligned} & (1, 2, 3, 4) \times (2, 3, 3, 1) \times (0, 2, 4, 6) \\ =& (T(1, 0, 0, 0), T(0, 1, 0, 0), T(0, 0, 1, 0), T(0, 0, 0, 1)) \\ =& \left(\begin{vmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 3 & 1 \\ 0 & 2 & 4 & 6 \\ 1 & 0 & 0 & 0 \end{vmatrix}, \begin{vmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 3 & 1 \\ 0 & 2 & 4 & 6 \\ 0 & 1 & 0 & 0 \end{vmatrix}, \begin{vmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 3 & 1 \\ 0 & 2 & 4 & 6 \\ 0 & 0 & 1 & 0 \end{vmatrix}, \begin{vmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 3 & 1 \\ 0 & 2 & 4 & 6 \\ 0 & 0 & 0 & 1 \end{vmatrix} \right) \\ =& (-4, 10, -8, 2) \end{aligned}$$

3. If $<, >$ is a bilinear form on a vector space V of dimension n, and $\{\vec{v}_1, ..., \vec{v}_n\}$ is a set of n vectors in V such that $< \vec{v}_i, \vec{v}_j >= \delta_{ij}$, prove $\{\vec{v}_1, ..., \vec{v}_n\}$ forms a basis of V.

   Solution: Since V is of dimension n, we only need to show $\{\vec{v}_1, ..., \vec{v}_n\}$ are linearly independent.

   If $a_1\vec{v}_1 + ... + a_n\vec{v}_n = \vec{0}$, then for any $\vec{v}_i$,

$$0 =< \vec{0}, \vec{v}_i >=< a_1\vec{v}_1 + ... + a_n\vec{v}_n, \vec{v}_i >= a_i$$

   So $a_1 = ... = a_n = 0$, we finish the proof.

4. If $<, >$ is a symmetric and positive definite bilinear form on a vector space V, a linear transformation $f : V \longrightarrow V$ is called **self-adjoint** with respect to $< . >$ if $< \vec{u}, f(\vec{v}) >=< f(\vec{u}), \vec{v} >$ for any $\vec{u}, \vec{v} \in V$. If $\{\vec{v}_1, ..., \vec{v}_n\}$ is an orthonormal

basis of V with respect to $<,>$, and A is the matrix of f with respect to this basis, prove that A is a symmetric matrix.

Solution: If $A = (a_{ij})$ is the matrix of f with respect to this basis, then $f(\vec{v}_j) = \sum_{k=1}^{n} a_{kj}\vec{v}_k$ for each j.

$< \vec{v}_i, f(\vec{v}_j) >=< \vec{v}_i, \sum_{k=1}^{n} a_{kj}\vec{v}_k >= a_{ij}$

and $< f(\vec{v}_i), \vec{v}_j >=< \sum_{k=1}^{n} a_{ki}\vec{v}_k, \vec{v}_j >= a_{ji}$

Since the bilinear form is self-adjoint, $< \vec{v}_i, f(\vec{v}_j) >=< f(\vec{v}_i), \vec{v}_j >$, i.e. $a_{ij} = a_{ji}$, the matrix A is symmetric.

5. V is a vector space, W_1 and W_2 are vector subspaces of V such that $W_1 \cap W_2 = \{\vec{0}\}$ and $dim(V) = dim(W_1) + dim(W_2)$. If $\{\vec{a}_1, ..., \vec{a}_k\}$ is a basis for W_1 and $\{\vec{b}_1, ..., \vec{b}_l\}$ is a basis for W_2, prove $\{\vec{a}_1, ..., \vec{a}_k, \vec{b}_1, ..., \vec{b}_l\}$ is a basis for V.

   Solution: $dim(V) = dim(W_1) + dim(W_2) = k + l$, so we only need to show $\{\vec{a}_1, ..., \vec{a}_k, \vec{b}_1, ..., \vec{b}_l\}$ are linearly independent.

   If $\lambda_1\vec{a}_1 + ... + \lambda_k\vec{a}_k + \mu_1\vec{b}_1 + ... + \mu_l\vec{b}_l = \vec{0}$, then

$$\lambda_1\vec{a}_1 + ... + \lambda_k\vec{a}_k = -(\mu_1\vec{b}_1 + ... + \mu_l\vec{b}_l)$$

   The left side of the above equation is in W_1 while the right side of the equation is in W_2, and $W_1 \cap W_2 = \{\vec{0}\}$, we get

$$\lambda_1\vec{a}_1 + ... + \lambda_k\vec{a}_k = \vec{0}$$

   and

$$\mu_1\vec{b}_1 + ... + \mu_l\vec{b}_l = \vec{0}$$

   Since $\{\vec{a}_1, ..., \vec{a}_k\}$ is a basis for W_1 and $\{\vec{b}_1, ..., \vec{b}_l\}$ is a basis for W_2, we conclude $\lambda_1 = ... = \lambda_k = 0$ and $\mu_1 = ... = \mu_l = 0$, which implies $\{\vec{a}_1, ..., \vec{a}_k, \vec{b}_1, ..., \vec{b}_l\}$ are linearly independent.

6. V is a vector space. $f : V \longrightarrow V$ is a linear transformation. Define the **pullback** of f on $T^k(V)$ to be the map

$$f^* : T^k(V) \longrightarrow T^k(V)$$

   defined by: for any $T \in T^k(V)$, $f^*(T)$ is given by

$$f^*(T)(\vec{v}_1, ..., \vec{v}_k) = T(f\vec{v}_1, ..., f\vec{v}_k)$$

Prove:

(i). $f^* : T^k(V) \longrightarrow T^k(V)$ is a linear transformation

Solution: For any $T, S \in T^k(V)$ and $\lambda, \mu \in \vec{F}$,

$$\begin{aligned}
&f^*(\lambda T + \mu S)(\vec{v_1}, ..., \vec{v}_k) \\
=&(\lambda T + \mu S)(f(\vec{v_1}), ..., f(\vec{v}_k)) \\
=&\lambda T(f(\vec{v_1}), ..., f(\vec{v}_k)) + \mu S(f(\vec{v_1}), ..., f(\vec{v}_k)) \\
=&\lambda f^*(T)(\vec{v_1}, ..., \vec{v}_k) + \mu f^*(T)(\vec{v_1}, ..., \vec{v}_k) \\
=&(\lambda f^*(T) + \mu f^*(T))(\vec{v_1}, ..., \vec{v}_k)
\end{aligned}$$

So $f^*(\lambda T + \mu S) = \lambda f^*(T) + \mu f^*(T)$

(ii). If $g : V \longrightarrow V$ is another linear transformation, then $(g \circ f)^* = f^* \circ g^*$

Solution: For any $T \in T^k(V)$,

$$\begin{aligned}
&(g \circ f)^*(T)(\vec{v_1}, ..., \vec{v}_k) \\
=&T(g(f(\vec{v_1})), ..., g(f(\vec{v}_k))) \\
=&g^*(T)(f(\vec{v_1}), ..., f(\vec{v}_k)) \\
=&f^*(g^*(T))(\vec{v_1}, ..., \vec{v}_k) \\
=&(f^* \circ g^*)(T)(\vec{v_1}, ..., \vec{v}_k)
\end{aligned}$$

So $(g \circ f)^* = f^* \circ g^*$