1. $\{\vec{v}_1, ..., \vec{v}_n\}$ are $n$ linearly independent vectors in $\mathbb{R}^n$. If $\vec{v}.\vec{v}_i = 0$ for any $i \in \{1, 2, ..., n\}$, prove $\vec{v} = \vec{0}$

   **Solution**: Since $\{\vec{v}_1, ..., \vec{v}_n\}$ are $n$ linearly independent vectors in $\mathbb{R}^n$, they form a basis of $\mathbb{R}^n$. So each element of the standard basis $\{\vec{e}_1, ..., \vec{e}_n\}$ can be written as a linear combination of $\{\vec{v}_1, ..., \vec{v}_n\}$, i.e. there exists $a_{i1}, ..., a_{in}$ such that

   $$\vec{e}_i = a_{i1}\vec{v}_1 + ... + a_{in}\vec{v}_n$$

   Then $\vec{v}.\vec{e}_i = a_{i1}\vec{v}.\vec{v}_1 + ... + a_{in}\vec{v}.\vec{v}_n = 0 + ... + 0 = 0$ for $i = 1, ..., n$.

   Since $\{\vec{e}_1, ..., \vec{e}_n\}$ is an orthonormal basis of $\mathbb{R}^n$, there exists $b_1, ..., b_n \in \mathbb{R}$ such that $\vec{v} = b_1\vec{e}_1 + ... + b_n\vec{v}_n$.

   $0 = \vec{v}.\vec{e}_i = b_1\vec{e}_1.\vec{e}_i + .. + b_n\vec{e}_n.\vec{e}_n = b_i$ for all $i = 1, ..., n$, we conclude $\vec{v} = 0$.

2. $\vec{S}$ is an oriented surface with boundary a closed curve $C$ with orientation compatible with that of $\vec{S}$. $f(x, y, z)$ is a smooth scalar function defined in a region containing $\vec{S}$.

   (i). Prove that for any unit vector $\hat{u}$,

   $$\hat{u}.(\iint_S -\vec{\nabla} f \times\, d\vec{S}) = \hat{u}.(\oint_C f\, d\vec{r})$$

   (Hint: Apply Stokes' Theorem to $f\hat{u}$)

   **Solution**:

   By Stokes' Theorem, we know

   $$\iint_S \vec{\nabla} f \times \hat{u}.\, d\vec{s} = \oint_C f\hat{u}.\, d\vec{r}$$

   so

   $$\iint_S \hat{u} \times (-\vec{\nabla} f)\, d\vec{S} = \oint_C f\hat{u}.\, d\vec{r}$$

   Since $\vec{u}$ is a constant vector, we see

   $$\hat{u}.(\iint_S -\vec{\nabla} f \times\, d\vec{S}) = \hat{u}.(\oint_C f\, d\vec{r})$$

(ii). Prove $\iint_S -\vec{\nabla} f \times d\vec{S} = \oint_C f\, d\vec{r}$

**Solution**: We have shown in (i) that $\hat{u}.(\iint_S -\vec{\nabla} f \times d\vec{S}) = \hat{u}.(\oint_C f\, d\vec{r})$ for any unit vector $\vec{u}$, so

$$\vec{e}_i.(\iint_S -\vec{\nabla} f \times d\vec{S} - \oint_C f\, d\vec{r}) = 0$$

for the standard basis $\{\vec{e}_1, ..., \vec{e}_n\}$. By Question 1, we conclude

$$\iint_S -\vec{\nabla} f \times d\vec{S} - \oint_C f\, d\vec{r} = 0$$

So

$$\iint_S -\vec{\nabla} f \times d\vec{S} = \oint_C f\, d\vec{r}$$

3. $f, g, h$ are single-variable smooth functions $\mathbb{R} \longrightarrow \mathbb{R}$. Show that the vector field $\vec{F}(x, y, z) = (f(x), g(y), h(z))$ is conservative.

   **Solution**: $\vec{F}$ is a vector field defined on $\vec{R}^3$, which is open and simply-connected. We see $\vec{\nabla} \times \vec{F} = \vec{\nabla} \times (f(x), g(y), h(z)) = 0$, so the vector field is conservative.

4. Use Stokes' Theorem to evaluate $\iint_S \vec{\nabla} \times \vec{F}. d\vec{S}$ where $\vec{F} = (y^2 z, zx, x^2 y^2)$, and $\vec{S}$ is the part of the paraboloid $z = x^2 + y^2$ that lies inside the cylinder $x^2 + y^2 = 1$, oriented upward.

   **Solution**: The boundary of $\vec{S}$ is the circle $x^2 + y^2 = 1, z = 1$ oriented counter-clockwise, which can be parameterized by $\vec{r}(t) = (\cos t, \sin t, 1), 0 \leq t \leq 2\pi$. So by Stokes' Theorem,

$$\begin{aligned}
&\iint_S \vec{\nabla} \times \vec{F}. d\vec{S} \\
=&\oint_C \vec{F}. d\vec{r} \\
=&\int_0^{2\pi} (\sin^2 t, \cos t, \cos^2 \sin^2 t).(-\sin t, \cos t, 0)\, dt \\
=&\int_0^{2\pi} -\sin^3 t + \cos^2 t\, dt \\
=&\pi
\end{aligned}$$

5. $S$ is the cylinder $x^2+y^2=1, 0 \leq z \leq 1$. The two boundary circles $C_1$ and $C_2$ are oriented counterclockwise seen from above. If $\vec{F}$ is a vector field on $\mathbb{R}^3$ such that $\iint_S \vec{\nabla} \times \vec{F}. d\vec{S} = 0$, prove

$$\int_{C_1} \vec{F}. d\vec{r} = \int_{C_2} \vec{F}. d\vec{r}$$

**Solution**: Let $S_1$ be the disk $x^2+y^2=1, z=1$ and $S_2$ be the disk $x^2+y^2=1, z=0$. $S \cup S_1 \cup S_2$ is a closed surface, and we take the outward orientation of it. Then

$$\begin{aligned}
&\int_{S \cup S_1 \cup S_2} \vec{\nabla} \times \vec{F}. d\vec{S} = 0 \\
\Rightarrow &\int_S \vec{\nabla} \times \vec{F}. d\vec{S} + \int_{S_1} \vec{\nabla} \times \vec{F}. d\vec{S} + \int_{S_2} \vec{\nabla} \times \vec{F}. d\vec{S} = 0 \\
\Rightarrow &0 + \int_{C_1} \vec{F}. d\vec{r} + \int_{-C_2} \vec{F}. d\vec{r} = 0 \\
\Rightarrow &\int_{C_1} \vec{F}. d\vec{r} = \int_{C_2} \vec{F}. d\vec{r}
\end{aligned}$$