1. $\vec{F}(x, y, z) = (y, z, x)$ is a vector field. $\vec{S}$ is the unit sphere $x^2 + y^2 + z^2 = 1$ with outward orientation. Compute

$$\iint_S \vec{F} \times d\vec{S}$$

Solution: The sphere is parameterized by $\vec{r}(\phi, \theta) = (\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi)$, which agrees with the outward orientation.

$$\begin{aligned}
&\iint_S \vec{F} \times d\vec{S} \\
=&\iint_D \vec{F}(\vec{r}) \times (\frac{\partial\vec{r}}{\partial\phi} \times \frac{\partial\vec{r}}{\partial\theta})\, dA \\
=&\iint_D (\sin\phi\sin\theta, \cos\phi, \sin\phi\cos\theta) \times (\sin^2\phi\cos\theta, \sin^2\phi\sin\theta, \sin\phi\cos\phi)\, dA \\
=&(\int_0^{2\pi}\int_0^{\pi} \sin\phi\cos^2\phi - \sin^3\phi\sin\theta\cos\theta\, d\phi\, d\theta, \\
&\int_0^{2\pi}\int_0^{\pi} \sin^3\phi\cos^2\theta - \sin^2\phi\cos\phi\sin\theta\, d\phi\, d\theta, \\
&\int_0^{2\pi}\int_0^{\pi} \sin^3\phi\sin^2\theta - \sin^2\phi\cos\phi\cos\theta\, d\phi\, d\theta) \\
=&(\frac{4\pi}{3}, \frac{4\pi}{3}, \frac{4\pi}{3})
\end{aligned}$$

2. $\vec{F}(x, y, z) = x + y + z$ is a scalar function. C is the oriented intersection curve of the paraboloid $z = x^2 + y^2$ and the plane $x + y = 0$ from $(0, 0, 0)$ to $(1, -1, 2)$. Compute

$$\int_C f\, d\vec{r}$$

Solution:

The curve can be parameterized by $\vec{r}(t) = (t, -t, 2t^2)$, $0 \le t \le 1$.

$$\begin{aligned}
&\int_C f\,d\vec{r}\\
=&\int_0^1 f(\vec{r}(t))(\vec{r}(t))'\,dt\\
=&\int_0^1 2t^2(1,-1,4t)\,dt\\
=&\int_0^1 (2t^2,-2t^2,8t^3)\,dt\\
=&(\int_0^1 2t^2\,dt,\int_0^1 -2t^2\,dt,\int_0^1 8t^3\,dt)\\
=&(\frac{2}{3},-\frac{2}{3},2)
\end{aligned}$$

3. If $f(x,y,z)$ is a scalar function and $\vec{F}(x,y,z)$ is a vector field, prove

$$\vec{\nabla}\times(f\vec{F})=f(\vec{\nabla}\times\vec{F})+\vec{\nabla}f\times\vec{F}$$

Solution: Let $\vec{F}=(P,Q,R)$

$$\begin{aligned}
&\vec{\nabla}\times(f\vec{F})\\
=&\vec{\nabla}\times(fP,fQ,fR)\\
=&((fR)_y-(fQ)_z,(fP)_z-(fR)_x,(fQ)_x-(fP)_y)\\
=&f(R_y-Q_z,P_z-R_x,Q_x-P_y)+(f_yR-f_zQ,f_zP-f_xR,f_xQ-f_yP)\\
=&f(R_y-Q_z,P_z-R_x,Q_x-P_y)+(f_x,f_y,f_z)\times(P,Q,R)\\
=&f(\vec{\nabla}\times\vec{F})+\vec{\nabla}f\times\vec{F}
\end{aligned}$$

4. If $f(x,y,z)$ and $g(x,y,z)$ are scalar functions, $\vec{F}=g\vec{\nabla}f$, show that

$$\vec{F}.Curl(\vec{F})=0$$

Solution: By Problem 3, we have

$$\vec{\nabla}\times(g\vec{\nabla}f)=g(\vec{\nabla}\times\vec{\nabla}f)+\vec{\nabla}g\times\vec{\nabla}f=\vec{\nabla}g\times\vec{\nabla}f$$

So

$$(g\vec{\nabla}f).(\vec{\nabla}\times(g\vec{\nabla}f))=g\vec{\nabla}f.(\vec{\nabla}g\times\vec{\nabla}f)=0$$

5. Show that $\vec{F}(x, y, z) = (y, z, x)$ is not a conservative vector field.

   Solution:

   $\vec{\nabla} \times \vec{F} = (-1, -1, -1) \neq \vec{0}$, so the vector field is not conservative.