1. Find the surface area of the torus obtained from rotating the circle in yz-plane $(y-b)^2+z^2=a^2$ around z-axis, where $0<a<b$ are constants

   Solution: The circle $(y-b)^2+z^2=a^2$ can be parameterized as $y=b+a\cos u$, $z=a\sin u$, $0\le u\le 2\pi$. So the torus can be parameterized by:

   $$\vec{r}(u,v)=((b+a\cos u)\cos v,(b+a\cos u)\sin v,a\sin u),0\le u\le 2\pi,0\le v\le 2\pi$$

   $\frac{\partial\vec{r}}{\partial u}=(-a\sin u\cos v,-a\sin u\sin v,a\cos u)$
   $\frac{\partial\vec{r}}{\partial v}=(-(b+a\cos u)\sin v,(b+a\cos u)\cos v,0)$
   $\frac{\partial\vec{r}}{\partial u}\times\frac{\partial\vec{r}}{\partial v}=(-a(b+a\cos u)\cos u\cos v,-a(b+a\cos u)\cos u\sin v,-a(b+a\cos u)\sin u)$
   $|\frac{\partial\vec{r}}{\partial u}\times\frac{\partial\vec{r}}{\partial v}|=a(b+a\cos u)$

   $$\begin{aligned}\iint_S 1 &= \iint_D |\frac{\partial\vec{r}}{\partial u}\times\frac{\partial\vec{r}}{\partial v}|\,dA\\ &= \int_0^{2\pi}\int_0^{2\pi} a(b+a\cos u)\,du\,dv\\ &= 4\pi^2 ab\end{aligned}$$

2. Evaluate

   $$\iint_S x^2z+y^2z\,dS$$

   where S is the hemisphere $x^2+y^2+z^2=4$, $z\ge 0$.

   Solution: The surface is parameterized by

   $$\vec{r}(\rho,\theta)=(2\sin\phi\cos\theta,2\sin\phi\sin\theta,2\cos\phi),0\le\phi\le\frac{\pi}{2},0\le\theta\le 2\pi$$

$$
\begin{aligned}
&\iint_S x^2z + y^2z\, dS \\
=&\iint_D [(2\sin\phi\cos\theta)^2(2\cos\phi) + (2\sin\phi\sin\theta)^2(2\cos\phi)]|\frac{\partial\vec{r}}{\partial\phi} \times \frac{\partial\vec{r}}{\partial\theta}|\, dA \\
=&\int_0^{2\pi}\int_0^{\frac{\pi}{2}} (8\sin^2\phi\cos\phi)(4\sin\phi)\, d\phi\, d\theta \\
=&16\pi
\end{aligned}
$$

3. A surface S is parameterized by $\vec{r}(\theta, z) = (\cos\theta, \sin\theta, z)$, $0 \le \theta \le 2\pi, -1 \le z \le 1$. It is orientation agrees with $\frac{\partial\vec{r}}{\partial\theta} \times \frac{\partial\vec{r}}{\partial z}$

   (i). What surface is it geometrically?

   Solution: It is a cylinder

   (ii). If $\vec{F}(x, y, z) = (-y, x, z)$, evaluate

$$\iint_S \vec{F}.d\vec{S}$$

   Solution:
   $\frac{\partial\vec{r}}{\partial\theta} \times \frac{\partial\vec{r}}{\partial z} = (-\sin\theta, \cos\theta, 0) \times (0, 0, 1) = (\cos\theta, \sin\theta, 0)$

$$
\begin{aligned}
\iint_S \vec{F}.d\vec{S} &= \iint_D (-\sin\theta, \cos\theta, z).(\frac{\partial\vec{r}}{\partial\theta} \times \frac{\partial\vec{r}}{\partial z})\, dA \\
&= \iint_D (-\sin\theta, \cos\theta, z).(\cos\theta, \sin\theta, 0)\, dA \\
&= \iint_D 0\, dA \\
&= 0
\end{aligned}
$$

   (iii). Can you explain the result you obtained in (ii) geometrically?

   Solution: At each point (x, y, z) of this surface, its normal direction is parallel to the vector $(x, y, 0)$, and $\vec{F}(x, y, z) = (-y, x, z) \perp (x, y, z)$. So at each point, the vector field is parallel to the tangent plane, so no flux through the surface.

4. If S is the part of the graph of a smooth function $z = f(x, y) = 1 - x^2 - y^2$ above xy-plane with upward orientation, $\vec{F}(x, y, z) = (y, x, z)$, compute

$$\iint_S \vec{F}.d\vec{S}$$

Solution:

$$\begin{aligned}
\iint_S \vec{F}.d\vec{S} &= \iint_D -y\frac{\partial f}{\partial x} - x\frac{\partial f}{\partial y} + (1 - x^2 - y^2)\, dA \\
&= \iint_D 4xy + 1 - x^2 - y^2\, dA \\
&= \int_0^{2\pi}\int_0^1 (4r^2 \cos\theta \sin\theta + 1 - r^2) r\, dr\, d\theta \\
&= \frac{\pi}{2}
\end{aligned}$$

5. Evaluate the surface integral $\iint_S \vec{F}.\, d\vec{S}$, where $\vec{F} =< x, -z, y >$ and S is the part of the sphere $x^2 + y^2 + z^2 = 4$ in the first octant, with orientation pointing towards the origin.

Solution: The surface is parameterized by

$$\vec{r}(\rho, \theta) = (2 \sin\phi \cos\theta, 2 \sin\phi \sin\theta, 2 \cos\phi), 0 \le \phi \le \frac{\pi}{2}, 0 \le \theta \le \frac{\pi}{2}$$

$$\frac{\partial\vec{r}}{\partial\phi} \times \frac{\partial\vec{r}}{\partial\theta} = (4 \sin^2\phi \cos\theta, 4 \sin^2 \sin\theta, 4 \sin\phi \cos\phi)$$

and we know $\frac{\partial\vec{r}}{\partial\phi} \times \frac{\partial\vec{r}}{\partial\theta}$ gives the outward orientation, so

$$
\begin{aligned}
&\iint_S \vec{F}.\, d\vec{S} \\
&= -\int_0^{2\pi}\int_0^{2\pi} (2\sin\phi\cos\theta, -2\cos\phi, 2\sin\phi\sin\theta).(\frac{\partial\vec{r}}{\partial\phi}\times\frac{\partial\vec{r}}{\partial\theta})\, d\phi\, d\theta \\
&= -\int_0^{2\pi}\int_0^{2\pi} (2\sin\phi\cos\theta, -2\cos\phi, 2\sin\phi\sin\theta).(4\sin^2\phi\cos\theta, 4\sin^2\sin\theta, 4\sin\phi\cos\phi)\, d\phi\, d\theta \\
&= -\int_0^{2\pi}\int_0^{2\pi} 8\sin^3\phi\cos^2\theta\, d\phi\, d\theta \\
&= -\frac{4\pi}{3}
\end{aligned}
$$