1. Compute $\int_C \vec{F}.d\vec{r}$, where $\vec{F} = (-y, x)$, and C is the closed path on the circle $x^2 + y^2 = R^2$ along counterclockwisely direction.

   Solution: The circle can be parameterized by $\vec{r}(t) = (R\cos t, R\sin t)$, $t \in [0, 2\pi]$.

$$\begin{aligned}\int_C \vec{F}.d\vec{r} &= \int_0^{2\pi} (-R\sin t, R\cos t).(-R\sin t, R\cos t)\, dt \\ &= \int_0^{2\pi} R^2(\sin^2 t + \cos^2 t)\, dt \\ &= 2\pi R^2\end{aligned}$$

2. If C is a smooth curve in $\mathbb{R}^2$ parameterized by $\vec{r}(t), a \le t \le b$, and $\vec{v}$ is a constant vector, show that

$$\int_C \vec{v} \cdot d\vec{r} = \vec{v} \cdot [\vec{r}(b) - \vec{r}(a)]$$

   Solution: Let $\vec{v} = (v_1, v_2)$, $\vec{r}(t) = (x(t), y(t))$

$$\begin{aligned}\int_C \vec{v} \cdot d\vec{r} &= \int_a^b \vec{v}.\vec{r}(t)'\, dt \\ &= \int_a^b v_1 x'(t) + v_2 y'(t)\, dt \\ &= v_1 \int_a^b x'(t)\, dt + v_2 \int_a^b y'(t)\, dt \\ &= v_1(x(b) - x(a)) + v_2(y(b) = y(a)) \\ &= \vec{v}.[\vec{r}(b) - \vec{r}(a)]\end{aligned}$$

3. Find a potential function of the vector field

$$\vec{F}(x, y) = (xy^2, x^2y)$$

Solution: Assume $\vec{F}(x,y) = \nabla f$, then we get

$$\begin{cases} xy^2 = \frac{\partial f}{\partial x} \\ x^2y = \frac{\partial f}{\partial y} \end{cases}$$

By the first equation we get $f(x,y) = \frac{x^2y^2}{2} + g(y)$, then we see

$$\frac{\partial f}{\partial y} = x^2y + g'(y)$$

So $g'(y) = 0$, which implies $g(y) = C$ for some constant C. The potential function $f(x,y) = \frac{x^2y^2}{2} + C$ for some constant C.

4. (i). Show that the line integral is independent of path

$$\int_C 2xe^{-y}\,dx + (2y - x^2e^{-y})\,dy$$

C is any path from $(1,0)$ to $(2,0)$.

Solution: The vector field $\vec{F} = (2xe^{-y}, 2y - x^2e^{-y})$ is defined on $\mathbb{R}$, which is open and simply-connected, so we can check:
$\frac{\partial}{\partial x}(2y - x^2e^{-y}) = -2xe^{-y} = \frac{\partial}{\partial y}(2xe^{-y})$
So the vector field is conservative, hence the line integral is independent of path.

(ii). Evaluate the above integral

Solution: Since it is conservative, we are going to find a potential function: $\vec{F} = \nabla f$:

$$\begin{cases} 2xe^{-y} = \frac{\partial f}{\partial x} \\ 2y - x^2e^{-y} = \frac{\partial f}{\partial y} \end{cases}$$

By the first equation we get $f(x,y) = x^2e^{-y} + g(y)$, then take $\frac{\partial}{\partial y}$, we get $\frac{\partial f}{\partial y} = -x^2e^{-y} + g'(y)$.

So $g'(y) = 2y$, $g(y) = y^2 + C$ for some constant C, and we may take $C = 0$, to get $f(x,y) = x^2e^{-y} + y^2$.

$\int_C 2xe^{-y}\,dx + (2y - x^2e^{-y})\,dy = f(2,0) - f(1,0) = 4 - 1 = 3$

5. $\vec{F}(x,y) = (P(x,y), Q(x,y))$ is a conservative vector field defined on $\mathbb{R}^2$ such that P and Q are smooth functions, and $\vec{F} \neq (0,0)$ for all points. If $g : \mathbb{R}^2 \longrightarrow \mathbb{R}$ is a smooth function, prove $g\vec{F}$ is conservative if and only if ∇g is parallel to $\vec{F}$ everywhere.

   Solution: $\vec{F}$ is conservative, so $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$.

   Since the plane is open and simply-connected, $g\vec{F} = (gP, gQ)$ is conservative if and only if
$$\frac{\partial(gP)}{\partial y} = \frac{\partial(gQ)}{\partial x}$$
   if and only if
$$\frac{\partial g}{\partial y}P + g\frac{\partial P}{\partial y} = \frac{\partial g}{\partial x}Q + g\frac{\partial Q}{\partial x}$$
   if and only if
$$\frac{\partial g}{\partial y}P = \frac{\partial g}{\partial x}Q$$
   if and only if
$$(-\frac{\partial g}{\partial y}, \frac{\partial g}{\partial x}).\vec{F} = 0$$
   if and only if $\vec{F} \perp (-\frac{\partial g}{\partial y}, \frac{\partial g}{\partial x})$ if and only if $\vec{F}$ is parallel to ∇g, since $\nabla g \perp (-\frac{\partial g}{\partial y}, \frac{\partial g}{\partial x})$.