1. $F : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ and $G : \mathbb{R}^m \longrightarrow \mathbb{R}^k$ are differentiable maps. Prove the chain rule:
$$D(G \circ F)(x) = DG(F(x))DF(x)$$

Solution:

Let $(y_1, ..., y_m) = F(x_1, ..., x_n)$ and $(z_1, ..., z_k) = G(y_1, ..., y_m)$.

$$\begin{aligned}
&[(D(G \circ F))(x_1, ..., x_n)]_{ij} \\
=&\frac{\partial z_i}{\partial x_j}(x_1, ..., x_n) \\
=&\frac{\partial z_i}{\partial y_l}(y_1, ..., y_m)\frac{\partial y_l}{\partial x_j}(x_1, ..., x_n) \\
=&[(DG)(y_1, ..., y_m)DF(x_!, ..., x_n)]_{ij} \\
=&[(DG)(F(x_1, ..., x_n))DF(x_!, ..., x_n)]_{ij}
\end{aligned}$$

So we get

$$(D(G \circ F))(x_1, ..., x_n) = (DG)(F(x_1, ..., x_n))DF(x_1, ..., x_n)$$

i.e.
$$(D(G \circ F))(x) = (DG)(F(x))DF(x)$$

2. (1). $f : \mathbb{R} \longrightarrow \mathbb{R}$ is a differentiable function such that $f'(x) \neq 0$ for any $x \in \mathbb{R}$. Prove f is a one-to-one function.

Solution:

For any $x_1 \neq x_2$, by the mean value theorem, there s c between x_1 and x_2 such that $f(x_1) - f(x_2) = f'(c)(x_1 - x_2) \neq 0$, so $f(x_1) \neq f(x_2)$.

(ii). Prove $p : \mathbb{R} \longrightarrow \mathbb{R}$ defined by $p(x) = x^5 + x^3 + x + 1$ is a bijective function.

Solution:

$p'(x) = 5x^4 + 3x^2 + 1 > 0$, so p is injective.

Observe that $\lim\limits_{x\to+\infty} p(x) = +\infty$ and $\lim\limits_{x\to-\infty} p(x) = -\infty$, so for any $y \in \mathbb{R}$, we can find x_1 such that $p(x_1) > y$ and x_2 such that $p(x_2) < y$. By the Intermediate Value Theorem, there exists x between x_1 and x_2 such that $p(x) = y$, so p is surjective.

(iii). Compute $(p^{-1})'(1)$

Solution:

Note that $p(0) = 1$, so $(p^{-1})'(1) = \frac{1}{p'(0)} = 1$

3. Verify that the function $F(x, y) = (x^2 + y^2, x - y^3)$ is locally invertible at $(0, 1)$, and compute $DF^{-1}(1, -1)$

Solution:

$DF(x, y) = \begin{bmatrix} 2x & 2y \\ 1 & -3y^2 \end{bmatrix}$, so $DF(0, 1) = \begin{bmatrix} 0 & 2 \\ 1 & -3 \end{bmatrix}$, which is invertible since its determinant is -2. So F is localy invertible at $(0, 1)$.

Note $F(0, 1) = (1, -1)$, so $DF^{-1}(1, -1) = DF(0, 1)^{-1} = \begin{bmatrix} \frac{3}{2} & 1 \\ \frac{1}{2} & 0 \end{bmatrix}$

4. $F : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a continuously differentiable function. If there is a sequence $\{x_n\} \in \mathbb{R}^n$ such that $\lim\limits_{x\to\infty} x_n = x_0$, $x_0 \neq x_n$ for any $n \in \mathbb{N}$, and $f(x_n) = a \in \mathbb{R}^n$ for any $n \in \mathbb{N}$, prove $DF(x_0)$ is not invertible.

Solution:

$F(x_0) = f(\lim_{x\to\infty} x_n) = \lim_{x\to\infty} f(x_n) = a$.

Suppose $DF(x_0)$ is invertible, then by the Inverse Function Theorem, there is neighbourhood V of z_0 and neighbourhood W of a such that $F : V \longrightarrow W$ is invertible map. But $\{x_n\} \in \mathbb{R}^n$, so we can find some x_N such that $x_N \in V$. But $x_N \neq x_0$ and $F(x_N) = F(x_0) = a$, contradict to $F : V \longrightarrow W$ being invertible, contradiction.

5. $F(x, y, z) = (x + y + z, x^2 + y^2 + z^2)$. Prove there exist differentiable $g(x)$ and $h(x)$ defined in some neighbourhood of $x = 0$ such that $g(0) = 1, h(0) = 2$ and on this neighbourhood $F(x, g(x), h(x)) = (3, 5)$. Compute $g'(0)$ and $h'(0)$.

Solution:

$DF(x, y, z) = \begin{bmatrix} 1 & 1 & 1 \\ 2x & 2y & 2z \end{bmatrix}$ So $DF(0, 1, 2) = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 4 \end{bmatrix}$

Note that $DF(0,1,2) = \begin{bmatrix} 1 & 1 \\ 2 & 4 \end{bmatrix}$ is invertible, so by the Implicit Function Theorem, there is a neighbourhood V of 0 in $\mathbb{R}$ and $G : V \longrightarrow \mathbb{R}^2$ given by $G(x) = (g(x), h(x))$ such that $F(x, G(x)) = F(0,1,2) = (3,5)$, i.e. $F(x, g(x), h(x)) = (3,5)$

By Implicit Function Theorem,

$$\begin{bmatrix} g'(0) \\ h'(0) \end{bmatrix} = -\begin{bmatrix} 1 & 1 \\ 2 & 4 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

6. A surface is defined by the equation $xy + y^2z + z^3x = 3$. Find the equation of the tangent plane for the surface at $(1,1,1)$.

   Solution:

   Let $F(x,y,z) = xy + y^2z + z^3x - 3$.

   $\nabla F(x,y,z) = (y + z^3, x + 2yz, y^2 + 3z^2x)$, so $\nabla F(1,1,1) = (2,3,4)$.

   The tangent plane is

$$2(x-1) + 3(x-1) + 4(x-1) = 0$$