1. \( F : \mathbb{R}^n \to \mathbb{R}^m \) and \( G : \mathbb{R}^m \to \mathbb{R}^k \) are differentiable maps. Prove the chain rule:
   \[ D(G \circ F)(x) = DG(F(x))DF(x) \]

   **Solution:**
   Let \((y_1, \ldots, y_m) = F(x_1, \ldots, x_n)\) and \((z_1, \ldots, z_k) = G(y_1, \ldots, y_m)\).
   
   \[
   \left[ (D(G \circ F))(x_1, \ldots, x_n) \right]_{ij} = \frac{\partial z_i}{\partial x_j}(x_1, \ldots, x_n) \\
   = \frac{\partial z_i}{\partial y_l}(y_1, \ldots, y_m) \frac{\partial y_l}{\partial x_j}(x_1, \ldots, x_n) \\
   = \left[ (DG)(y_1, \ldots, y_m)DF(x_1, \ldots, x_n) \right]_{ij} \\
   = \left[ (DG)(F(x_1, \ldots, x_n))DF(x_1, \ldots, x_n) \right]_{ij}
   \]
   
   So we get
   \[
   (D(G \circ F))(x_1, \ldots, x_n) = (DG)(F(x_1, \ldots, x_n))DF(x_1, \ldots, x_n)
   \]
   i.e.
   \[
   (D(G \circ F))(x) = (DG)(F(x))DF(x)
   \]

2. (1). \( f : \mathbb{R} \to \mathbb{R} \) is a differentiable function such that \( f'(x) \neq 0 \) for any \( x \in \mathbb{R} \). Prove \( f \) is a one-to-one function.

   **Solution:**
   For any \( x_1 \neq x_2 \), by the mean value theorem, there's \( c \) between \( x_1 \) and \( x_2 \) such that \( f(x_1) - f(x_2) = f'(c)(x_1 - x_2) \neq 0 \), so \( f(x_1) \neq f(x_2) \).

   (ii). Prove \( p : \mathbb{R} \to \mathbb{R} \) defined by \( p(x) = x^5 + x^3 + x + 1 \) is a bijective function.

   **Solution:**
   \( p'(x) = 5x^4 + 3x^2 + 1 > 0 \), so \( p \) is injective.
Observe that \( \lim_{x \to +\infty} p(x) = +\infty \) and \( \lim_{x \to -\infty} p(x) = -\infty \), so for any \( y \in \mathbb{R} \), we can find \( x_1 \) such that \( p(x_1) > y \) and \( x_2 \) such that \( p(x_2) < y \). By the Intermediate Value Theorem, there exists \( x \) between \( x_1 \) and \( x_2 \) such that \( p(x) = y \), so \( p \) is surjective.

(iii). Compute \((p^{-1})'(1)\)

**Solution:**

Note that \( p(0) = 1 \), so \((p^{-1})'(1) = \frac{1}{p'(0)} = 1\)

3. Verify that the function \( F(x, y) = (x^2 + y^2, x - y^3) \) is locally invertible at \((0, 1)\), and compute \( DF^{-1}(1, -1) \)

**Solution:**

\[
DF(x, y) = \begin{bmatrix} 2x & 2y \\ 1 & -3y^2 \end{bmatrix}, \text{ so } DF(0, 1) = \begin{bmatrix} 0 & 2 \\ 1 & -3 \end{bmatrix}, \text{ which is invertible since its determinant is } -2. \text{ So } F \text{ is locally invertible at } (0, 1).
\]

Note \( F(0, 1) = (1, -1) \), so \( DF^{-1}(1, -1) = DF(0, 1)^{-1} = \begin{bmatrix} \frac{3}{2} & 1 \\ \frac{1}{2} & 0 \end{bmatrix} \)

4. \( F : \mathbb{R}^n \to \mathbb{R}^n \) is a continuously differentiable function. If there is a sequence \( \{x_n\} \in \mathbb{R}^n \) such that \( \lim_{x \to \infty} x_n = x_0, x_0 \neq x_n \) for any \( n \in \mathbb{N} \), and \( f(x_n) = a \in \mathbb{R}^n \) for any \( n \in \mathbb{N} \), prove \( DF(x_0) \) is not invertible.

**Solution:**

\( F(x_0) = f(\lim_{x \to \infty} x_n) = \lim_{x \to \infty} f(x_n) = a. \)

Suppose \( DF(x_0) \) is invertible, then by the Inverse Function Theorem, there is a neighbourhood \( V \) of \( v_0 \) and neighbourhood \( W \) of \( a \) such that \( F : V \to W \) is invertible map. But \( \{x_n\} \in \mathbb{R}^n \), so we can find some \( x_N \) such that \( x_N \in V \). But \( x_N \neq x_0 \) and \( F(x_N) = F(x_0) = a \), contradict to \( F : V \to W \) being invertible, contradiction.

5. \( F(x, y, z) = (x + y + z, x^2 + y^2 + z^2) \). Prove there exist differentiable \( g(x) \) and \( h(x) \) defined in some neighbourhood of \( x = 0 \) such that \( g(0) = 1, h(0) = 2 \) and on this neighbourhood \( F(x, g(x), h(x)) = (3, 5) \). Compute \( g'(0) \) and \( h'(0) \).

**Solution:**

\[
DF(x, y, z) = \begin{bmatrix} 1 & 1 & 1 \\ 2x & 2y & 2z \end{bmatrix} \text{ So } DF(0, 1, 2) = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 4 \end{bmatrix}
\]
Note that $DF(0, 1, 2) = \begin{bmatrix} 1 & 1 \\ 2 & 4 \end{bmatrix}$ is invertible, so by the Implicit Function Theorem, there is a neighbourhood $V$ of $0$ in $\mathbb{R}$ and $G : V \rightarrow \mathbb{R}^2$ given by $G(x) = (g(x), h(x))$ such that $F(x, G(x)) = F(0, 1, 2) = (3, 5)$, i.e. $F(x, g(x), h(x)) = (3, 5)$.

By Implicit Function Theorem,

$$\begin{bmatrix} g'(0) \\ h'(0) \end{bmatrix} = -\begin{bmatrix} 1 & 1 \\ 2 & 4 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

6. A surface is defined by the equation $xy + y^2z + z^3x = 3$. Find the equation of the tangent plane for the surface at $(1, 1, 1)$.

**Solution:**

Let $F(x, y, z) = xy + y^2z + z^3x - 3$.

$\nabla F(x, y, z) = (y + z^3, x + 2yz, y^2 + 3z^2x)$, so $\nabla F(1, 1, 1) = (2, 3, 4)$.

The tangent plane is

$$2(x - 1) + 3(y - 1) + 4(z - 1) = 0$$