1. If T_{ij} represents a 2-tensor, prove T_{ii} is independent of coordinates.

   Solution:

   $T'_{i'j'} = a_{i'i}a_{j'j}T_{ij}$, so

$$T'_{i'i'} = a_{i'i}a_{i'j}T_{ij} = (A^tA)_{ij}T_{ij} = \delta_{ij}T_{ij} = T_{ii}$$

2. If T_i and S_j represent 1-tensors, prove T_iS_j represents a 2-tensor.

   Solution:

   $T'_{i'} = a_{i'i}T_i$ and $S'_{j'} = a_{j'j}S_j$ since they are 1-tensors.

$$T'_{i'}S'_{j'} = a_{i'i}T_i a_{j'j}S_j = a_{i'i}a_{j'j}T_iS_j$$

   So it represents a 2-tensor.

3. If T_{ijkl} represents a tensor of rank 4, prove T_{ijjl} represents a tensor of rank 2.

   Solution:

   $T'_{i'j'k'l'} = a_{i'i}a_{j'j}a_{k'k}a_{l'l}T_{ijkl}$, so

$$T'_{i'j'j'l'} = a_{i'i}a_{j'j}a_{j'k}a_{l'l}T_{ijkl} = a_{i'i}a_{l'l}(A^tA)_{jk}T_{ijkl} = a_{i'i}a_{l'l}\delta_{jk}T_{ijkl} = a_{i'i}a_{l'l}T_{ijjl}$$

4. Construct an isotropic 5-tensor on $\mathbb{R}^3$ and prove it is isotropic.

   Solution: Claim $T_{ijklm} = \delta_{ij}\epsilon_{klm}$ is an isotropic 5-tensor.

$$\begin{aligned}
T'_{i'j'k'l'm'} &= a_{i'i}a_{j'j}a_{k'k}a_{l'l}a_{m'm}T_{ijklm} \\
&= a_{i'i}a_{j'j}a_{k'k}a_{l'l}a_{m'm}\delta_{ij}\epsilon_{klm} \\
&= (a_{i'i}a_{j'j}\delta_{ij})(a_{k'k}a_{l'l}a_{m'm}\epsilon_{klm}) \\
&= \delta'_{i'j'}\epsilon'_{k'l'm'} \\
&= \delta_{i'j'}\epsilon_{k'l'm'} \\
&= T_{i'j'k'l'm'}
\end{aligned}$$

5. (i). If x_i represents a 1-tensor, and $A = (a_{ij})$ denotes the matrix of change of basis, prove that
$$\frac{\partial x_i'}{\partial x_j} = a_{ij} \text{ and } \frac{\partial x_i}{\partial x_j'} = a_{ji}$$

Solution:

We know $x_i' = a_{ik}x_k$, so
$$\frac{\partial x_i'}{\partial x_j} = \frac{\partial(a_{ik}x_k)}{\partial x_j} = a_{ik}\frac{\partial x_k}{\partial x_j} = a_{ik}\delta_{jk} = a_{ij}$$
Similarly, $x_i = a_{ji}x_j'$, so
$$\frac{\partial x_i}{\partial x_j'} = \frac{\partial(a_{ki}x_k')}{\partial x_j'} = a_{ki}\frac{\partial x_k'}{\partial x_j'} = a_{ki}\delta_{jk} = a_{ji}$$

(ii). If f is a smooth function, prove $\frac{\partial f}{\partial x_i}$ represents a 1-tensor.

Solution:
$$\frac{\partial f}{\partial x_{i'}'} = \frac{\partial f}{\partial x_i}\frac{\partial x_i}{\partial x_{i'}'} = a_{i'i}\frac{\partial f}{\partial x_i}$$

(iii). If u_i represents a 1-tensor, prove $\frac{\partial u_i}{\partial x_j}$ represents a 2-tensor.

Solution:
$$\frac{\partial u_{i'}'}{\partial x_{j'}'} = \frac{\partial(a_{i'i}u_i)}{\partial x_{j'}'} = a_{i'i}\frac{\partial u_i}{\partial x_j}\frac{\partial x_j}{\partial x_{j'}'} = a_{i'i}a_{j'j}\frac{\partial u_i}{\partial x_j}$$

6. $\sigma \in S_n$. Define the $n \times n$ matrix $A = (\delta_{i\sigma^{-1}(j)})$.

(i). If $n = 3$, $\sigma(1) = 3, \sigma(2) = 1, \sigma(3) = 2$, write out A explicitly.

Solution:
$$A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

(ii). Prove $A \in O_n(\mathbb{R})$.

Solution:

$$(A^tA)_{ij} = A^t_{ik}A_{kj} = A_{ki}A_{kj} = \delta_{\sigma^{-1}(k)i}\delta_{\sigma^{-1}(k)j} = \delta_{ij}$$

So $A^tA = I_n$, the identity matrix, we get $A^{-1} = A^t$, $A \in O_n\mathbb{R}$

(iii). Prove $A \in SO_n(\mathbb{R})$ if and only if $sgn(\sigma) = +1$.

Solution: We need to show $\det A = 1$ if and only if $sgn(\sigma) = +1$:

$$\begin{aligned}
\det A &= \sum_{\tau \in S_n} sgn(\tau)\delta_{1\sigma^{-1}(\tau(1))}...\delta_{n\sigma^{-1}(\tau(n))} \\
&= \sum_{\tau \in S_n} sgn(\sigma\tau)\delta_{1\sigma^{-1}(\sigma\tau(1))}...\delta_{n\sigma^{-1}(\sigma\tau(n))} \\
&= sgn(\sigma)\sum_{\tau \in S_n} sgn(\tau)\delta_{1\tau(1)}...\delta_{n\tau(n)} \\
&= sgn(\sigma)\det I_n \\
&= sgn(\sigma)
\end{aligned}$$

We see $\det A = 1$ if and only if $sgn(\sigma) = +1$