1. Show that if $\vec{u}+\vec{v}$ and $\vec{u}-\vec{v}$ are orthogonal, then $|\vec{u}|=|\vec{v}|$

   Solution:

   Method I:

   If $\vec{u}+\vec{v} \perp \vec{u}-\vec{v}$, then

$$\begin{aligned}(\vec{u}+\vec{v})\cdot(\vec{u}-\vec{v}) &= 0\\ \vec{u}\cdot\vec{u}-\vec{u}\cdot\vec{v}+\vec{v}\cdot\vec{u}-\vec{v}\cdot\vec{v} &= 0\\ \vec{u}\cdot\vec{u}-\vec{v}\cdot\vec{v} &= 0\\ |\vec{u}|^2-|\vec{v}|^2 &= 0\\ |\vec{u}|^2 &= |\vec{v}|^2\\ |\vec{u}| &= |\vec{v}|\end{aligned}$$

   Method II:

   Assume $\vec{u}=< x_1, y_1, z_1 >$, $\vec{v}=< x_2, y_2, z_2 >$.

$$\begin{aligned}(\vec{u}+\vec{v})\cdot(\vec{u}-\vec{v}) &= 0\\ < x_1+x_2, y_1+y_2, z_1+z_2 > \cdot < x_1-x_2, y_1-y_2, z_1-z_2 > &= 0\\ (x_1+x_2)(x_1-x_2)+(y_1+y_2)(y_1-y_2)+(z_1+z_2)(z_1-z_2) &= 0\\ x_1^2-x_2^2+y_1^2-y_2^2+z_1^2-z_2^2 &= 0\\ x_1^2+y_1^2+z_1^2 &= x_2^2+y_2^2+z_2^2\\ |\vec{u}|^2 &= |\vec{v}|^2\\ |\vec{u}| &= |\vec{v}|\end{aligned}$$

2. Find an unit vector that makes an angle of $\frac{\pi}{3}$ with $\vec{v}=< 1, \sqrt{3}, -2\sqrt{3} >$ and perpendicular to $\vec{k}=< 0, 0, 1 >$.

   Solution: Assume $\vec{u}=< x, y, z >$ is a unit vector ($|\vec{u}|=1$) that makes an angle of $\frac{\pi}{3}$ with $\vec{v}$ and perpendicular to $\vec{k}=< 0, 0, 1 >$, then

$$0 =< x, y, z > \cdot < 0, 0, 1 >= z$$

so $\vec{u} =< x, y, 0 >$. Also,

$$\frac{1}{2} = \cos\frac{\pi}{3} = \frac{\vec{u}\cdot\vec{v}}{|\vec{u}||\vec{v}|} = \frac{x+\sqrt{3}y}{4}$$

We get $x + \sqrt{3}y = 2$. Together with $|\vec{u}| = x^2 + y^2 = 1$, we get $x = \frac{1}{2}, y = \frac{\sqrt{3}}{2}$, so $\vec{u} =< \frac{1}{2}, \frac{\sqrt{3}}{2}, 0 >$

3. $\vec{u}\cdot\vec{v} = \sqrt{3}$ and $\vec{u}\times\vec{v} =< 1, 2, 2 >$. Compute the angle between $\vec{u}$ and $\vec{v}$.

   Solution: Let θ be the angle between $\vec{u}$ and $\vec{v}$.

   $\sqrt{3} = \vec{u}\cdot\vec{v} = |\vec{u}||\vec{v}|\cos\theta$

   $3 = |\vec{u}\times\vec{v}| = |\vec{u}||\vec{v}|\sin\theta$

   So the quotient of the above equations implies $\cot\theta = \frac{\cos\theta}{\sin\theta} = \frac{\sqrt{3}}{3}$, and $0 \le \theta \le \pi$, hence $\theta = \frac{\pi}{3}$

4. Find the distance from $(1, 2, 4)$ to the plane $3x + 2y + z - 5 = 0$

   Solution: Let $P = (1, 2, 4)$. We can arbitrarily pick a point on the plane, say $Q = (0, 0, 5)$. Then $\overrightarrow{PQ} =< -1, -2, 1 >$. By the equation of the plane, we see $\vec{n} =< 3, 2, 1 >$ is a normal vector to the plane.

   So the distance from P to the plane is $\frac{|\overrightarrow{PQ}\cdot\vec{n}|}{|\vec{n}|} = \frac{|-6|}{\sqrt{14}} = \frac{3}{7}\sqrt{14}$

5. Find an equation of the plane passing through $(0, 2, 4)$, $(1, -3, 2)$ and $(-3, -2, 1)$

   Solution: Let $P = (0, 2, 4)$, $Q = (1, -3, 2)$, $R = (-3, -2, 1)$.

   Then $\overrightarrow{PQ} =< 1, -5, -2 >$ and $\overrightarrow{PR} =< -3, -4, -3 >$ are parallel to the plane, so $\overrightarrow{PQ}\times\overrightarrow{PR} =< 7, 9, -19 >$ is a normal vector to the plane. So the equation of the plane is given by

   $7(x - 0) + 9(y - 2) - 19(z - 4) = 0$, i.e. $7x + 9y - 19z + 58 = 0$