An important application of dot product is in geometry. We can describe a plane in \( \mathbb{R}^3 \) by an equation that involves dot product:

Pick a vector \( \mathbf{n} = (a, b, c) \) perpendicular to the plane \( \alpha \).

Pick a point \( P_0 = (x_0, y_0, z_0) \in \alpha \).

Now for any \( P = (x, y, z) \), \( P \in \alpha \) if and only if \( \overrightarrow{P_0P} \perp \mathbf{n} \). (We call \( \mathbf{n} \) a normal vector of \( \alpha \)).

Algebraically, \( \overrightarrow{P_0P} \perp \mathbf{n} \) is equivalent to

\[
(a-x_0, b-y_0, c-z_0) \cdot (a, b, c) = 0.
\]

i.e. \( a(x-x_0) + b(y-y_0) + c(z-z_0) = 0 \).

or \( ax + by + cz = ax_0 + by_0 + cz_0 \).

Example The plane that passes through \((1, 2, 3)\) with a normal vector \((5, -2, 1)\) has equation

\[
5(x-1) - 2(x-2) + (x-1) = 0.
\]

The dot product assigns a real number (i.e. scalar) to a given pair of vectors, as we've seen in its definition. Now we're going to consider another form of vector product, called cross product.
Definition. \( \vec{u} \) and \( \vec{v} \) are vectors in \( \mathbb{R}^3 \). Define the cross product \( \vec{u} \times \vec{v} \) to be the vector whose magnitude is \( |\vec{u}| |\vec{v}| \sin \theta \), whose direction is determined by the "right hand rule". \( \theta \) is the angle between \( \vec{u} \) and \( \vec{v} \).

**Example.**

\[ \vec{u} \times \vec{v} \] is a vector perpendicular to both \( \vec{u} \) and \( \vec{v} \).

**Proposition.** \( \vec{u}, \vec{v}, \vec{w} \) are vectors in \( \mathbb{R}^3 \)

(i) \( \vec{u} \times \vec{v} = -\vec{v} \times \vec{u} \)

(ii) \( \vec{u} \times \vec{0} = \vec{0} \)

(iii) \( \vec{u} \times \vec{u} = \vec{0} \)

(iv) \( \vec{u} \parallel \vec{v} \Rightarrow \vec{u} \times \vec{v} = \vec{0} \)

(v) \( |\vec{u} \times \vec{v}| \) equals to the area of the parallelogram determined by \( \vec{u} \) and \( \vec{v} \)

(vi) \( \vec{u} \times (\vec{v} + \vec{w}) = \vec{u} \times \vec{v} + \vec{u} \times \vec{w} \)

**Proof.**

(i) By the right hand rule, if the order of the two vectors is switched, the direction of the cross product will be changed to the opposite.

(ii) \( |\vec{u} \times \vec{0}| = |\vec{u}| |\vec{0}| \sin \theta = 0 \), so \( \vec{u} \times \vec{0} = \vec{0} \)

(iii) by (i), let \( \vec{v} = \vec{u} \), then \( \vec{u} \times \vec{u} = -\vec{u} \times \vec{u} \), so \( \vec{u} \times \vec{u} = \vec{0} \)
(iv) If \( \vec{u} \parallel \vec{v} \), then \( \theta = 0 \) or \( \theta = \pi \), so \( \sin \theta = 0 \).

\[ |\vec{u} \times \vec{v}| = |\vec{u}| \cdot |\vec{v}| \sin \theta = 0. \]

(v) By the Sine Theorem, the area of the parallelogram is

\[ 2 \cdot \frac{|\vec{u}| \cdot |\vec{v}| \sin \theta}{2} = |\vec{u} \times \vec{v}|. \]

(vi) To prove (vi), we need a Lemma:

**Lemma.** The cross product of \( \vec{u} \) and \( \vec{v} \) only depends on the component of \( \vec{u} \) perpendicular to \( \vec{v} \).

This Lemma can be verified by the following picture:

We decompose \( \vec{v} = \overrightarrow{a} + \overrightarrow{b} \) such that \( \overrightarrow{a} \perp \vec{u} \) and \( \overrightarrow{b} \parallel \vec{u} \).

Then \( \vec{u} \times \overrightarrow{v} = \vec{u} \times \overrightarrow{a} \).

Now with the help of this Lemma, we can prove (vi):

\[ \vec{u} \times (\vec{v} + \vec{w}) = \vec{u} \times \vec{v} + \vec{u} \times \vec{w}. \]

Let \( \vec{v} = \overrightarrow{a} + \overrightarrow{b}, \vec{w} = \overrightarrow{c} + \overrightarrow{d} \), such that

\[ \overrightarrow{a} \perp \vec{u}, \overrightarrow{b} \parallel \vec{u}, \overrightarrow{c} \parallel \vec{u}, \overrightarrow{d} \perp \vec{u}. \]

By the Lemma, we know that
\( \vec{u} \times (\vec{v} + \vec{w}) = \vec{u} \times (\vec{a} + \vec{z} + \vec{c} + \vec{d}) = \vec{u} \times ((\vec{a} + \vec{c}) + (\vec{d} + \vec{z})) = \vec{u} \times (\vec{a} + \vec{c}) \) 

\[ \vec{u} \times \vec{v} + \vec{u} \times \vec{w} = \vec{u} \times \vec{a} + \vec{u} \times \vec{c} \] 

So it suffices to verify \( \vec{u} \times (\vec{a} + \vec{c}) = \vec{u} \times \vec{a} + \vec{u} \times \vec{c} \) for \( \vec{a} \perp \vec{u} \), \( \vec{c} \parallel \vec{u} \). This can be done as follows:

Consider \( \vec{u} \) to be pointing into the page, so \( \vec{\alpha} \) and \( \vec{c} \) are parallel to the page.

\( \vec{u} \times \vec{\alpha} \) is the vector of magnitude \( 1\vec{u} \cdot 1\vec{\alpha} \), in the direction of \( \vec{\alpha} \) rotating clockwise \( 90^\circ \),

\( \vec{u} \times \vec{c} \) is the vector of magnitude \( 1\vec{u} \cdot 1\vec{c} \), in the direction of \( \vec{c} \) rotating clockwise \( 90^\circ \).

\( \vec{u} \times (\vec{\alpha} + \vec{c}) \) is the vector of magnitude \( 1\vec{u} \cdot 1\vec{\alpha} + 1\vec{c} \), in the direction of \( \vec{\alpha} + \vec{c} \) rotating clockwise \( 90^\circ \).

Then geometrically we see very clearly that \( \vec{u} \times (\vec{\alpha} + \vec{c}) \) coincides with \( \vec{u} \times \vec{\alpha} + \vec{u} \times \vec{c} \).
Now with the help of the distributive law, we can develop an algebraic way of computation.

Again we take the standard basis \( \{ \overrightarrow{i}, \overrightarrow{j}, \overrightarrow{k} \} \).

Observe that \( \overrightarrow{i} \times \overrightarrow{j} = \overrightarrow{k}, \overrightarrow{j} \times \overrightarrow{k} = \overrightarrow{i}, \overrightarrow{k} \times \overrightarrow{i} = \overrightarrow{j} \).

If \( \overrightarrow{u} = (x_1, y_1, z_1), \overrightarrow{v} = (x_2, y_2, z_2) \), then
\[
\overrightarrow{u} = x_1 \overrightarrow{i} + y_1 \overrightarrow{j} + z_1 \overrightarrow{k}, \quad \overrightarrow{v} = x_2 \overrightarrow{i} + y_2 \overrightarrow{j} + z_2 \overrightarrow{k}.
\]

\[
\overrightarrow{u} \times \overrightarrow{v} = (x_1 \overrightarrow{i} + y_1 \overrightarrow{j} + z_1 \overrightarrow{k}) \times (x_2 \overrightarrow{i} + y_2 \overrightarrow{j} + z_2 \overrightarrow{k})
\]
\[
= (x_1 \overrightarrow{i} \times (y_2 \overrightarrow{k})) + (x_1 \overrightarrow{j} \times (z_2 \overrightarrow{k})) + (y_1 \overrightarrow{i} \times (z_2 \overrightarrow{k}))
\]
\[
+ (y_1 \overrightarrow{j} \times (z_2 \overrightarrow{k})) + (z_1 \overrightarrow{k} \times (z_2 \overrightarrow{k})) + (z_1 \overrightarrow{k} \times (y_2 \overrightarrow{j})).
\]

An interesting observation is that above indicates:
\[
\overrightarrow{u} \times \overrightarrow{v} = \begin{vmatrix}
\overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\
x_1 & y_1 & z_1 \\
x_2 & y_2 & z_2
\end{vmatrix}
\]

The cross product also has applications in geometry: it can be used to describe straight lines in \( \mathbb{R}^3 \).

Given a straight line \( l \subset \mathbb{R}^3 \), it can be described as \( \overrightarrow{r} = \overrightarrow{a} + \lambda \overrightarrow{u} \), where \( \overrightarrow{a} \) is the position vector of a point \( A = (a_1, a_2, a_3) \) on \( l \), and \( \overrightarrow{u} \parallel l \).

\( \lambda \) is the parameter.
We wish to obtain a form of equation without the parameter \( \lambda \), so we use the cross product:

\[
\vec{r} \times \vec{u} = (\vec{a} + \lambda \vec{u}) \times \vec{u} = \vec{a} \times \vec{u}
\]

i.e. \( \vec{r} \times \vec{u} = \vec{a} \times \vec{u} \)

Note \( \vec{a} \) and \( \vec{u} \) are some fixed vectors, it follows \( \vec{a} \times \vec{u} = \vec{b} \) is some fixed vector.

So the equation of \( l \) can be written as

\[
\vec{r} \times \vec{u} = \vec{b}
\]

Now we are going to see another product of vectors: Scalar Triple Product.

Definition. Given three vectors \( \vec{u}, \vec{v}, \vec{w} \) in \( \mathbb{R}^3 \), their scalar triple product is \( \vec{u} \cdot (\vec{v} \times \vec{w}) \).

If \( \vec{u} = (u_1, u_2, u_3), \vec{v} = (v_1, v_2, v_3), \vec{w} = (w_1, w_2, w_3) \)

then

\[
\vec{u} \cdot (\vec{v} \times \vec{w}) = \begin{vmatrix}
    u_1 & u_2 & u_3 \\
    v_1 & v_2 & v_3 \\
    w_1 & w_2 & w_3 
\end{vmatrix}
\]
Proposition. (i) \( \vec{u} \cdot (\vec{v} \times \vec{w}) = \vec{v} \cdot (\vec{w} \times \vec{u}) = \vec{w} \cdot (\vec{u} \times \vec{v}) \)

(ii) \( \vec{u} \cdot (\vec{v} \times \vec{w}) = (\vec{u} \times \vec{v}) \cdot \vec{w} \)

(iii) \( \vec{u} \cdot (\vec{v} \times \vec{w}) \neq 0 \) if and only if \( \vec{u}, \vec{v}, \vec{w} \) are linearly independent.

(iv) \( \vec{u} \cdot (\vec{v} \times \vec{w}) \) is the signed volume of the parallelepiped formed by \( \vec{u}, \vec{v}, \vec{w} \).

Proof. The proofs follow easily from elementary properties of determinants.