1. If $T \in \Lambda^k(V)$, $\vec{v}_1, ..., \vec{v}_k$ is a set of k linearly dependent vectors on V, prove $T(\vec{v}_1, ..., \vec{v}_k) = 0$

2. If $T \in \Lambda^k(V)$ and $S \in \Lambda^l(V)$, prove $T \wedge S = (-1)^{kl} S \wedge T$

3. If $\{\vec{v}_1, ..., \vec{v}_n\}$ is a basis for a n-dimensional vector space V and $\{\phi_1, ..., \phi_n\}$ is the dual basis for V^*, prove for each $\sigma \in S_n$,

$$\phi_{\sigma(1)} \wedge ... \wedge \phi_{\sigma(n)} = sgn(\sigma)\phi_1 \wedge ... \wedge \phi_n$$

4. $T \in T^k(V)$ is called a symmetric k-tensor if for any $\vec{v}_1, ..., \vec{v}_k \in V$ and any $\sigma \in S_k$, $T(\vec{v}_{\sigma(1)}, ..., \vec{v}_{\sigma(k)}) = T(\vec{v}_1, ..., \vec{v}_k)$. The set of all symmetric k-tensors form the subspace $Sym^k(V)$ of $T^k(V)$.

(i). Given any $T \in T^k(V)$, define $Sym(T)$ to be the k-tensor:

$$Sym(T)(\vec{v}_1, ..., \vec{v}_k) = \frac{1}{k!} \sum_{\sigma \in S_k} T(\vec{v}_{\sigma(1)}, ..., \vec{v}_{\sigma(k)})$$

Prove $Sym(T) \in Sym^k(V)$.

(ii). If $T \in Sym^k(V)$, prove $Sym(T) = T$

(iii). $T \in T^2(V)$, prove $T = Sym(T) + Alt(T)$

(iv). Prove $Sym^2(V) \cap \Lambda^2(V) = \{0\}$

(v). $T \in T^2(V)$. Prove T can be decomposed as the sum of a symmetric 2-tensor and an alternating 2-tensor in a unique way. (Remark: This is equivalent to say $T^2(V) = Sym^2(V) \oplus \Lambda^2(V)$)

(vi). $\{\vec{u}_1, ..., \vec{u}_n\}$ is a basis of V. $T \in T^2(V)$. $A = (a_{ij})$ is the $n \times n$ matrix such that $a_{ij} = T(\vec{u}_i, \vec{u}_j)$. A square matrix A is called skew-symmetric if $A^t = -A$, where A^t denotes the transpose of A. Prove $T \in Sym^2(V)$ if and only if A is a symmetric matrix, and $T \in \Lambda^2(V)$ if and only if A is a skew-symmetric matrix.

(vii). Prove Each $n \times n$ matrix can be written as a sum of a symmetric matrix and a skew-symmetric matrix in a unique way.

(viii). Write the matrix $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ as the sum of a symmetric matrix and a skew-symmetric matrix.