1. Given $\vec{a} = (a_1, a_2, a_3), \vec{b} = (b_1, b_2, b_3) \in \mathbb{R}^3$, define a map

$$T : \mathbb{R}^3 \longrightarrow \mathbb{R}$$
$$\vec{v} = (x, y, z) \mapsto det \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ x & y & z \end{bmatrix}$$

(i). Prove $T \in T^1(\mathbb{R}^3)$

(ii). In (i) you proved $T \in T^1(\mathbb{R}^3)$, so by our discussion in class, T can be represented by

$$T(\vec{x}) = \vec{z}.\vec{x}$$

for some unique $\vec{z} \in \mathbb{R}^3$. Prove $\vec{z} = \vec{a} \times \vec{b}$.

2. We can generalize the idea in (1) to define the cross product in $\mathbb{R}^n$:

Given $\vec{a}_1 = (a_{1,1}, ..., a_{1,n}), ..., \vec{a}_{n-1} = (a_{n-1,1}, ..., a_{n-1,n}) \in \mathbb{R}^n$, define the map

$$T : \mathbb{R}^n \longrightarrow \mathbb{R}$$
$$\vec{x} = (x_1, x_2, ..., x_n) \mapsto det \begin{bmatrix} \vec{a}_1 \\ ... \\ \vec{a}_{n-1} \\ \vec{x} \end{bmatrix} = det \begin{bmatrix} a_{1,1} & ... & a_{1,n} \\ ... & ... & ... \\ a_{n-1,1} & ... & a_{n-1,n} \\ x_1 & ... & x_n \end{bmatrix}$$

Similar to (1), $T \in T^1(\mathbb{R}^n)$ and there is unique $z \in \mathbb{R}^n$ such that $T(\vec{x}) = \vec{z}.\vec{x}$. Define $\vec{a}_1 \times ... \times \vec{a}_{n-1} = \vec{z}$.

Compute $(1, 2, 3, 4) \times (2, 3, 3, 1) \times (0, 2, 4, 6) \in \mathbb{R}^4$.

3. If $<,>$ is a bilinear form on a vector space V of dimension n, and $\{\vec{v}_1, ..., \vec{v}_n\}$ is a set of n vectors in V such that $< \vec{v}_i, \vec{v}_j >= \delta_{ij}$, prove $\{\vec{v}_1, ..., \vec{v}_n\}$ forms a basis of V.

4. If $<,>$ is a symmetric and positive definite bilinear form on a vector space V, a linear transformation $f : V \longrightarrow V$ is called **self-adjoint** with respect to $< . >$ if $< \vec{u}, f(\vec{v}) >=< f(\vec{u}), \vec{v} >$ for any $\vec{u}, \vec{v} \in V$. If $\{\vec{v}_1, ..., \vec{v}_n\}$ is an orthonormal basis of V with respect to $<,>$, and A is the matrix of f with respect to this basis, prove that A is a symmetric matrix.

5. V is a vector space, W_1 and W_2 are vector subspaces of V such that $W_1 \cap W_2 = \{\vec{0}\}$ and $dim(V) = dim(W_1) + dim(W_2)$. If $\{\vec{a}_1, ..., \vec{a}_k\}$ is a basis for W_1 and $\{\vec{b}_1, ..., \vec{b}_l\}$ is a basis for W_2, prove $\{a_1, ..., a_k, b_1, ..., b_l\}$ is a basis for V.

6. V is a vector space. $f : V \longrightarrow V$ is a linear transformation. Define the **pullback** of f on $T^k(V)$ to be the map

$$f^* : T^k(V) \longrightarrow T^k(V)$$

defined by: for any $T \in T^k(V)$, $f^*(T)$ is given by

$$f^*(T)(\vec{v}_1, ..., \vec{v}_k) = T(f\vec{v}_1, ..., f\vec{v}_k)$$

Prove:

(i). $f^* : T^k(V) \longrightarrow T^k(V)$ is a linear transformation

(ii). If $g : V \longrightarrow V$ is another linear transformation, then $(g \circ f)^* = f^* \circ g^*$