Definition. If \( \mathbf{F}(x,y,z) \) is a vector field defined in a neighbourhood of a point \((x_0, y_0, z_0)\), \( V \) is a region in \( \mathbb{R}^3 \) such that \((x_0, y_0, z_0)\) is an interior point of \( V \), with boundary of \( V \) to be a surface \( S \), outward oriented. Then define the curl of \( \mathbf{F} \) at \((x_0, y_0, z_0)\) to be the vector

\[
\text{Curl} \mathbf{F}(x_0, y_0, z_0) = -\lim_{\text{Vol}(V) \to 0} \frac{\int_S \mathbf{F} \times d\mathbf{S}}{\text{Vol}(V)}
\]

if the limit exists. We then also get a corresponding vector field \( \text{Curl} \mathbf{F} \) for the given vector field \( \mathbf{F} \).

Proposition. If \( \mathbf{F} \) and \( \mathbf{G} \) are vector fields, \( \lambda, \mu \in \mathbb{R} \), then

\[
\text{Curl} \left( \lambda \mathbf{F} + \mu \mathbf{G} \right) = \lambda \text{Curl} \mathbf{F} + \mu \text{Curl} \mathbf{G}.
\]

Proof. It follows directly from the definition of \( \text{Curl} \) and the fact that cross product is distributive.

Proposition. If \( f(x,y,z) \) is a scalar function and \( \mathbf{U} \) is a constant vector field, then \( \text{Curl} (f \mathbf{U}) = \nabla f \times \mathbf{U} \)

Proof. \( \text{Curl} (f \mathbf{U}) = -\lim_{\text{Vol}(V) \to 0} \frac{\int_S f \mathbf{U} \times d\mathbf{S}}{\text{Vol}(V)} = -\lim_{\text{Vol}(V) \to 0} \frac{\int_S f \mathbf{U} \times \mathbf{n} dS}{\text{Vol}(V)} = -\lim_{\text{Vol}(V) \to 0} \mathbf{U} \times \frac{\int_S f \mathbf{n} dS}{\text{Vol}(V)} = -\mathbf{U} \times \nabla f = \nabla f \times \mathbf{U} \)
Proposition. If \( \vec{F}(x,y,z) = (P(x,y,z), Q(x,y,z), R(x,y,z)) \) is a vector field, then \( \text{Curl} \vec{F} = \left( \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \) \( \tag{1} \)

Proof. \( \vec{F} = P \vec{i} + Q \vec{j} + R \vec{k} \), so by the previous propositions,

\[
\text{Curl} \vec{F} = \text{Curl} (P \vec{i} + Q \vec{j} + R \vec{k}) \\
= \text{Curl} (P \vec{i}) + \text{Curl} (Q \vec{j}) + \text{Curl} (R \vec{k}) \\
= \nabla P \times \vec{i} + \nabla Q \times \vec{j} + \nabla R \times \vec{k} \\
= \left( \frac{\partial P}{\partial x}, \frac{\partial P}{\partial y}, \frac{\partial P}{\partial z} \right) \times (1,0,0) + \left( \frac{\partial Q}{\partial x}, \frac{\partial Q}{\partial y}, \frac{\partial Q}{\partial z} \right) \times (0,1,0) + \\
\left( \frac{\partial R}{\partial x}, \frac{\partial R}{\partial y}, \frac{\partial R}{\partial z} \right) \times (0,0,1) \\
= \left( \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right)
\]

Remark. A good way for memorizing the above result is that

\[
\text{Curl} \vec{F} = \begin{vmatrix}
\vec{i} & \vec{j} & \vec{k} \\
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
P & Q & R
\end{vmatrix} = \nabla \times \vec{F}
\]

Proposition. \( \text{Curl} (\nabla f) = \vec{0} \) for any smooth function \( f \).

Proof. \( \text{Curl} (\nabla f) = \text{Curl} \left( \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right) \)

\[
= \left( \frac{\partial}{\partial y} \left( \frac{\partial f}{\partial z} \right) - \frac{\partial}{\partial z} \left( \frac{\partial f}{\partial y} \right), \frac{\partial}{\partial z} \left( \frac{\partial f}{\partial x} \right) - \frac{\partial}{\partial x} \left( \frac{\partial f}{\partial z} \right), \frac{\partial}{\partial x} \left( \frac{\partial f}{\partial y} \right) - \frac{\partial}{\partial y} \left( \frac{\partial f}{\partial x} \right) \right)
\]

Corollary. If \( \vec{F} \) is a conservative vector field, then \( \text{Curl} \vec{F} = \vec{0} \)