Vectors

Liming Pang

1 Basic Definitions

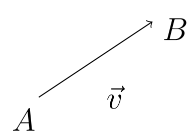
Definition 1. A vector in a line/plane/space is a quantity which has both magnitude and direction. The magnitude is a nonnegative real number and the direction is described by a ray in the line/plane/space. The magnitude of a vector $\vec{v}$ is dented by $|\vec{v}|$.

Remark 2. There is a unique vector of magnitude 0, denoted by $\vec{0}$, and we do not assign a specific direction to $\vec{0}$.

Example 3. *A good example of vectors is the concept of force in physics. When describing a force, we need to know its magnitude (how strong it is) and its direction. Forces of the same magnitude and different directions may have different effects on a particle/object.*

Example 4. *Each real number can be regarded as a vector. The magnitude of a real number is its absolute value, and the direction depends on whether it's positive, negative or zero.*

Geometric representation of vectors: An oriented line segment from a point A to a point B represents a vector $\vec{v}$. The length of the line segment represents the magnitude of $\vec{v}$, and the orientation represents the direction of $\vec{v}$. A is called the initial point of $\vec{v}$, and B is called the terminal point of $\vec{v}$. We also write $\vec{v} = \overrightarrow{AB}$.



With the help of geometric representation, we can define the algebraic operations of vectors.

Definition 5. $\vec{u}$ and $\vec{v}$ are two vectors, positioned in the way that the terminal point of $\vec{u}$ coincides with the initial point of $\vec{v}$, then define $\vec{u}+\vec{v}$ to be the vector with initial point same as the initial point of $\vec{u}$ and terminal point same as the terminal point of $\vec{v}$.

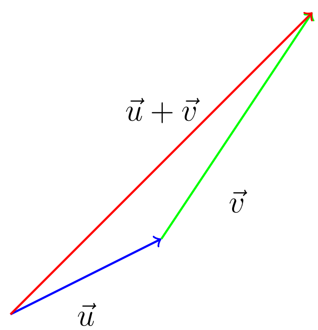


Figure 1: Vector Addition

Definition 6. $\vec{u}$ and $\vec{v}$ are two vectors, positioned in the way that the initial point of $\vec{u}$ coincides with the initial point of $\vec{v}$, then define $\vec{u}-\vec{v}$ to be the vector with initial point same as the terminal point of $\vec{v}$ and terminal point same as the terminal point of $\vec{u}$.

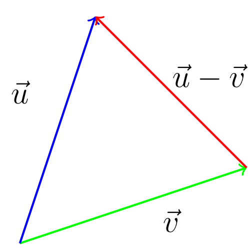


Figure 2: Vector Subtraction

Definition 7. $\vec{v}$ is a vector. Define $-\vec{v}$ to be the vector that has same magnitude with $\vec{v}$ but opposite direction.

Remark 8. By the above definitions, we can indeed interpret $\vec{u}-\vec{v}$ to be $\vec{u}+(-\vec{v})$, as shown by the following picture:

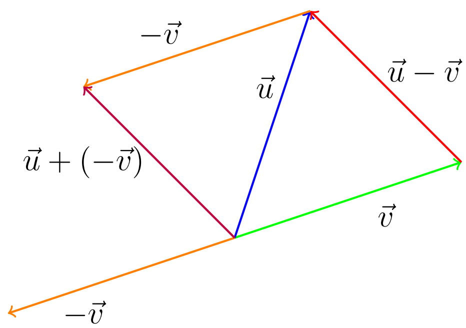


Figure 3: $\vec{u}-\vec{v}=\vec{u}+(-\vec{v})$

Definition 9. $\vec{\lambda}$ is a real number and $\vec{v}$ is a vector. Define the scalar multiplication $\lambda\vec{v}$ as follows: $|\lambda\vec{v}|=|\lambda||\vec{v}|$. The direction of $\lambda\vec{v}$ is same as that of $\vec{v}$ if $\lambda>0$, the direction of $\lambda\vec{v}$ is same as that of $-\vec{v}$ if $\lambda<0$, and $\lambda\vec{v}=\vec{0}$ if $\lambda=0$.

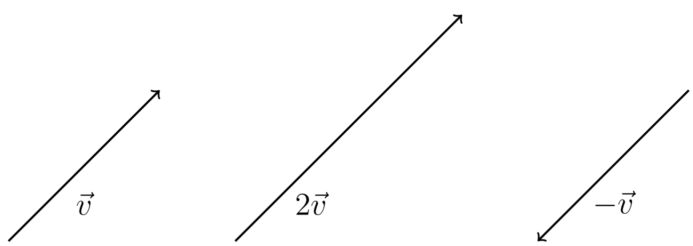


Figure 4: Scalar Multiplication

Remark 10. In particular, we see $-\vec{v}=(-1)\vec{v}$.

Definition 11. $\vec{u}$ and $\vec{v}$ are vectors forming an angle θ $(0\le\theta\le\pi)$ when their initial points coincide. Define the dot product of $\vec{u}$ and $\vec{v}$ to be the real number $\vec{u}.\vec{v}=|\vec{u}||\vec{v}|\cos\theta$

Example 12. *If we take the doc product of $\vec{v}$ with itself, $\theta=0$, we see*

$$\vec{v}.\vec{v}=|\vec{v}||\vec{v}|\cos 0=|\vec{v}|^2$$

Definition 13. Two nonzero vectors $\vec{u}$ and $\vec{v}$ are perpendicular if the rays representing their directions are perpendicular. Two nonzero vectors $\vec{u}$ and $\vec{v}$ are parallel if the rays representing their directions are parallel.

Proposition 14. *$\vec{u}$ and $\vec{v}$ are two nonzero vectors. They are perpendicular if and only if $\vec{u}.\vec{v} = 0$, and they are parallel if and only if $\vec{u}.\vec{v} = \pm|\vec{u}||\vec{v}|$*

Remark 15. When $\vec{u}$ is perpendicular to $\vec{v}$, we can denote it by $\vec{u} \perp \vec{v}$

Proposition 16. *The following rules hold for dot product:*

1. $\vec{u}.\vec{v} = \vec{v}.\vec{u}$
2. $(\lambda\vec{u}).\vec{v} = \vec{u}.(\lambda\vec{v}) = \lambda(\vec{u}.\vec{v})$
3. $\vec{u}.(\vec{v} + \vec{w}) = \vec{u}.\vec{v} + \vec{u}.\vec{w}$, $(\vec{v} + \vec{w}).\vec{u} = \vec{v}.\vec{u} + \vec{w}.\vec{u}$

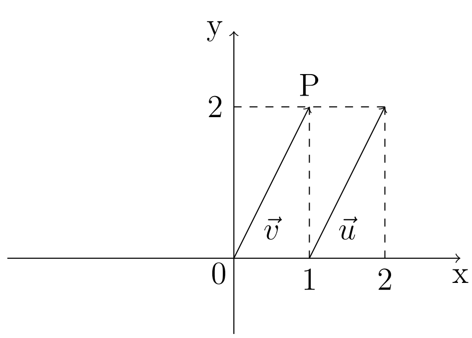
Exercise 17. *When $|\vec{u}| = |\vec{v}|$, show that $(\vec{u} + \vec{v}) \perp (\vec{u} - \vec{v})$ by applying the above propositions. Can you also give a geometric argument?*

2 Vectors in Cartesian Coordinates

In Cartesian Coordinates (i.e. rectangular coordinates), for any point $P = (x, y)$, we can construct the vector with initial point $0 = (0, 0)$ and terminal point P, and denote it as $\overrightarrow{OP}$. We call $\overrightarrow{OP}$ the position vector of the point P. On the other hand, given a vector $\vec{v}$, we put its initial point coincide with $0 = (0, 0)$ and its terminal point will be at some point $P = (x, y)$. We then denote $\vec{v} = \vec{OP} = (x, y)$.

Remark 18. We regard two vectors $\vec{u}$ and $\vec{v}$ to be equal if they have the same magnitude and same direction, regardless of their actual positions drawn on the plane.

Example 19. *In this example, $\vec{u} = \vec{v} = \overrightarrow{OP} = (1, 2)$*



The Coordinates bring a convenient description of vector operations:

Proposition 20. *If $\vec{u} = (x_1, y_1)$, $\vec{v} = (x_2, y_2)$, and λ is a real number, then $\vec{u} + \vec{v} = (x_1 + x_2, y_1 + y_2)$, $\vec{u} - \vec{v} = (x_1 - x_2, y_1 - y_2)$, $\lambda\vec{u} = (\lambda x_1, \lambda y_1)$.*

Proposition 21. *If $A = (x_1, y_1)$ and $B = (x_2, y_2)$, then the vector $\overrightarrow{AB} = (x_1 - x_2, y_1 - y_2)$.*

Proposition 22. *If $A = (x_1, y_1)$ and $B = (x_2, y_2)$, then the dot product $\vec{u}.\vec{v} = x_1x_2 + y_1y_2$*

Proposition 23. *If $\vec{v} = (x, y)$, then $|\vec{v}| = \sqrt{x^2 + y^2}$*

Definition 24. A vector $\vec{v}$ is called a unit vector if $|\vec{v}| = 1$.

Proposition 25. *If $\vec{v}$ is a nonzero vector, then the vector $\frac{1}{|\vec{v}|}\vec{v}$ is a unit vector that has the same direction as $\vec{v}$. We sometimes also write this vector as $\frac{\vec{v}}{|\vec{v}|}$.*

Example 26. *Let $A = (1, 3)$, $B = (2, 0)$. We see $\overrightarrow{AB} = (2, 0) - (1, 3) = (1, -3)$*

Example 27. *Let $\vec{u} = (1, 4)$, $\vec{v} = (4, -1)$. Then: $\vec{u} + \vec{v} = (1, 4) + (4, -1) = (5, 3)$, $\vec{u} - \vec{v} = (1, 4) - (4, -1) = (-3, 5)$. $\vec{u}.\vec{v} = 1 \times 4 + 4 \times (-1) = 0$, so $\vec{u} \perp \vec{v}$.*

Example 28. *$\vec{u} = (3, 4)$. $|\vec{u}| = \sqrt{3^2 + 4^2} = 5$, so $\frac{1}{5}\vec{u} = (\frac{3}{5}, \frac{4}{5})$ is a unit vector.*

The dot product can be applied to compute the angle between vectors: We know by definition $\vec{u}.\vec{v} = |\vec{u}||\vec{v}| \cos\theta$, so $\cos\theta = \frac{\vec{u}.\vec{v}}{|\vec{u}||\vec{v}|}$

Example 29. *$\vec{a} = (\sqrt{3}, 1)$, $\vec{b} = (3, -\sqrt{3})$. Let's compute the angle between these vectors. $\vec{a}.\vec{b} = \sqrt{3} \times 3 + 1 \times (-\sqrt{3}) = 2\sqrt{3}$. $|\vec{a}| = \sqrt{(\sqrt{3})^2 + 1^2} = 2$, $|\vec{b}| = \sqrt{3^2 + (\sqrt{3})^2} = 2\sqrt{3}$. So*

$$\cos\theta = \frac{\vec{a}.\vec{b}}{|\vec{a}||\vec{b}|} = \frac{2\sqrt{3}}{2 \times 2\sqrt{3}} = \frac{1}{2}$$

So $\cos\theta = \frac{1}{2}$ and $0 \leq \theta \leq \pi$, we conclude $\theta = \frac{\pi}{3}$