Homework V

Solution

1 First-Half

1. Compute

$$3\begin{bmatrix}-2 & 3 & 5\\ 1 & 7 & -1\end{bmatrix} - 5\begin{bmatrix}1 & 2 & -4\\ 0 & 3 & 2\end{bmatrix} + \begin{bmatrix}2 & 3 & 4\\ -2 & 1 & -5\end{bmatrix}$$

Solution:

$$\begin{aligned}
&3\begin{bmatrix}-2 & 3 & 5\\ 1 & 7 & -1\end{bmatrix} - 5\begin{bmatrix}1 & 2 & -4\\ 0 & 3 & 2\end{bmatrix} + \begin{bmatrix}2 & 3 & 4\\ -2 & 1 & -5\end{bmatrix}\\
=&\begin{bmatrix}-6 & 9 & 15\\ 3 & 21 & -3\end{bmatrix} - \begin{bmatrix}5 & 10 & -20\\ 0 & 15 & 10\end{bmatrix} + \begin{bmatrix}2 & 3 & 4\\ -2 & 1 & -5\end{bmatrix}\\
=&\begin{bmatrix}-9 & 2 & 39\\ 1 & 7 & -18\end{bmatrix}
\end{aligned}$$

2. Compute

$$\begin{bmatrix}-2 & 3 & 5\\ 1 & 2 & -1\end{bmatrix}\begin{bmatrix}2 & 1\\ 1 & 5\\ -1 & 3\end{bmatrix}\begin{bmatrix}3 & -1 & 5 & -2\\ 2 & -4 & 1 & 3\end{bmatrix}$$

Solution:

$$\begin{bmatrix} -2 & 3 & 5 \\ 1 & 2 & -1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 5 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 3 & -1 & 5 & -2 \\ 2 & -4 & 1 & 3 \end{bmatrix}$$
$$= \begin{bmatrix} -6 & 28 \\ 5 & 8 \end{bmatrix} \begin{bmatrix} 3 & -1 & 5 & -2 \\ 2 & -4 & 1 & 3 \end{bmatrix}$$
$$= \begin{bmatrix} 38 & -106 & -2 & 96 \\ 31 & -37 & 33 & 14 \end{bmatrix}$$

3. If A is a $n \times n$ square matrix, show that $A + A^T$ is a symmetric matrix.

   Solution:

   If $A = (a_{ij})$, then the (i, j) entry of $A + A^T$ is $a_{ij+a_{ji}}$, and the (i, j) entry of $(A + A^T)^T$ is the (j, i) entry of $A + A^T$, which is $a_{ji} + a_{ij} = a_{ij} + a_{ji}$. We conclude $A + A^T = (A + A^T)^T$, so $A + A^T$ is symmetric.

4. Use Gaussian Elimination to solve the following system of linear equations:

$$\begin{cases} 2x_2 + x_3 = -8 \\ x_1 - 2x_2 - 3x_3 = 0 \\ -x_1 + x_2 + 2x_3 = 3 \end{cases}$$

   Solution:

$$\left[\begin{array}{ccc|c} 0 & 2 & 1 & -8 \\ 1 & -2 & -3 & 0 \\ -1 & 1 & 2 & 3 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -2 & -3 & 0 \\ 0 & 2 & 1 & -8 \\ -1 & 1 & 2 & 3 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -2 & -3 & 0 \\ 0 & 2 & 1 & -8 \\ 0 & -1 & -1 & 3 \end{array}\right]$$
$$\sim \left[\begin{array}{ccc|c} 1 & -2 & -3 & 0 \\ 0 & 1 & \frac{1}{2} & -4 \\ 0 & -1 & -1 & 3 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -2 & -8 \\ 0 & 1 & \frac{1}{2} & -4 \\ 0 & 0 & -\frac{1}{2} & -1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -2 & -8 \\ 0 & 1 & \frac{1}{2} & -4 \\ 0 & 0 & 1 & 2 \end{array}\right]$$
$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & -4 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

   So the solution is $x_1 = -4, x_2 = -5, x_3 = 2$

5. Use Gaussian Elimination to solve the following system of linear equations:

$$\begin{cases} x_1 + 2x_2 - x_3 + 4x_4 = 4 \\ x_2 - x_3 + x_4 = 2 \\ 2x_1 + x_2 + x_3 - x_4 = 6 \end{cases}$$

Solution:

$$\left[\begin{array}{cccc|c} 1 & 2 & -1 & 4 & 4 \\ 0 & 1 & -1 & 1 & 2 \\ 2 & 1 & 1 & -1 & 6 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 2 & -1 & 4 & 4 \\ 0 & 1 & -1 & 1 & 2 \\ 0 & -3 & 3 & -9 & -2 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 1 & 2 & 0 \\ 0 & 1 & -1 & 1 & 2 \\ 0 & 0 & 0 & -6 & 4 \end{array}\right]$$
$$\sim \left[\begin{array}{cccc|c} 1 & 0 & 1 & 2 & 0 \\ 0 & 1 & -1 & 1 & 2 \\ 0 & 0 & 0 & 1 & -\frac{2}{3} \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & \frac{4}{3} \\ 0 & 1 & -1 & 0 & \frac{8}{3} \\ 0 & 0 & 0 & 1 & -\frac{2}{3} \end{array}\right]$$

So we get

$$\begin{cases} x_1 + x_3 = \frac{4}{3} \\ x_2 - x_3 = \frac{8}{3} \\ x_4 = -\frac{2}{3} \end{cases}$$

Let $x_3 = t$, we get the solutions are

$$\begin{cases} x_1 = \frac{4}{3} - t \\ x_2 = \frac{8}{3} + t \\ x_3 = t \\ x_4 = -\frac{2}{3} \end{cases}$$

where t can be any real number.

2 Second-Half

1. Find if the following matrix is invertible or not:

$$\begin{bmatrix} 1 & -3 & 2 \\ 3 & 0 & 4 \\ -2 & 5 & -1 \end{bmatrix}$$

Solution:

$$
\begin{aligned}
&\det\begin{bmatrix} 1 & -3 & 2 \\ 3 & 0 & 4 \\ -2 & 5 & -1 \end{bmatrix} \\
=&1 \times \det\begin{bmatrix} 0 & 4 \\ 5 & -1 \end{bmatrix} - 3 \times \det\begin{bmatrix} -3 & 2 \\ 5 & -1 \end{bmatrix} + (-2) \times \det\begin{bmatrix} -3 & 2 \\ 0 & 4 \end{bmatrix} \\
=&1 \times (-20) - 3 \times (-7) + (-2) \times (-12) \\
=&25
\end{aligned}
$$

2. Find the inverse of the following matrix by Gaussian Elimination:

$$
\begin{bmatrix} 2 & -1 & 3 \\ 0 & 5 & 8 \\ 4 & -2 & 1 \end{bmatrix}
$$

Solution:

$$
\left[\begin{array}{ccc|ccc} 2 & -1 & 3 & 1 & 0 & 0 \\ 0 & 5 & 8 & 0 & 1 & 0 \\ 4 & -2 & 1 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & -\frac{1}{2} & \frac{3}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 5 & 8 & 0 & 1 & 0 \\ 4 & -2 & 1 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & -\frac{1}{2} & \frac{3}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 5 & 8 & 0 & 1 & 0 \\ 0 & 0 & -5 & -2 & 0 & 1 \end{array}\right]
$$

$$
\sim \left[\begin{array}{ccc|ccc} 1 & -\frac{1}{2} & \frac{3}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 1 & \frac{8}{5} & 0 & \frac{1}{5} & 0 \\ 0 & 0 & -5 & -2 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & \frac{23}{10} & \frac{1}{2} & \frac{1}{10} & 0 \\ 0 & 1 & \frac{8}{5} & 0 & \frac{1}{5} & 0 \\ 0 & 0 & -5 & -2 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & \frac{23}{10} & \frac{1}{2} & \frac{1}{10} & 0 \\ 0 & 1 & \frac{8}{5} & 0 & \frac{1}{5} & 0 \\ 0 & 0 & 1 & \frac{2}{5} & 0 & -\frac{1}{5} \end{array}\right]
$$

$$
\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{21}{50} & 0 & \frac{23}{50} \\ 0 & 1 & 0 & -\frac{16}{25} & \frac{1}{5} & \frac{8}{25} \\ 0 & 0 & 1 & \frac{2}{5} & 0 & -\frac{1}{5} \end{array}\right]
$$

So the inverse matrix is

$$
\begin{bmatrix} -\frac{21}{50} & 0 & \frac{23}{50} \\ -\frac{16}{25} & \frac{1}{5} & \frac{8}{25} \\ \frac{2}{5} & 0 & -\frac{1}{5} \end{bmatrix}
$$

3. Solve the system of equations by Cramer's Rule:

$$\begin{cases} x + 3y - z = 2 \\ 2x + y + z = 3 \\ y - 2z = 1 \end{cases}$$

Solution:

$$A = \begin{bmatrix} 1 & 3 & -1 \\ 2 & 1 & 1 \\ 0 & 1 & -2 \end{bmatrix}, A_1 = \begin{bmatrix} 2 & 3 & -1 \\ 3 & 1 & 1 \\ 1 & 1 & -2 \end{bmatrix}, A_2 = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 3 & 1 \\ 0 & 1 & -2 \end{bmatrix}, A = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 1 & 3 \\ 0 & 1 & 1 \end{bmatrix},$$

$$\det A = 7, \det A_1 = 13, \det A_2 = -1, \det A_3 = -4$$

So

$$x_1 = \frac{\det A_1}{\det A} = \frac{13}{7}, x_2 = \frac{\det A_2}{\det A} = -\frac{1}{7}, x_3 = \frac{\det A_3}{\det A} = -\frac{4}{7}$$

4. Find values for the constants a and b such that the system of equations has a unique solution

$$\begin{cases} ax + y = 3 \\ x + z = 2 \\ y + az + bw = 6 \\ y + w = 1 \end{cases}$$

Solution:

$$A = \begin{bmatrix} a & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & a & b \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

$$\det A = ab - 2a = a(b-2)$$

The system of equations has unique solution if and only if $\det A \neq 0$, i.e. $a \neq 0$ and $b \neq 2$

5. $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$. Show that there is no 2×2 matrix B such that $A = B^2$

Solution:

$\det A = -2$. Suppose there is B such that $B^2 = A$, then

$$-2 = \det A = \det(B^2) = \det(B)^2$$

Contradiction. So there is no such matrix B.