Homework IV

Solution

1 First-Half

1. Compute
$$\int x^8\sqrt{x^3-1}\,dx$$

   Solution:

   Let $u = x^3 - 1$, then $du = (x^3-1)'dx = 3x^2dx$
$$\begin{aligned}\int x^8\sqrt{x^3-1}\,dx &= \frac{1}{3}\int (x^3)^2\sqrt{x^3-1}(3x^2)\,dx\\ &= \frac{1}{3}\int (u+1)^2u^{\frac{1}{2}}\,du\\ &= \frac{1}{3}\int u^{\frac{5}{2}} + 2u^{\frac{3}{2}} + u^{\frac{1}{2}}\,du\\ &= \frac{1}{3}(\frac{2}{7}u^{\frac{7}{2}} + 2\times\frac{2}{5}u^{\frac{5}{2}} + \frac{2}{3}u^{\frac{3}{2}}) + C\\ &= \frac{2}{21}(x^3-1)^{\frac{7}{2}} + \frac{4}{15}(x^3-1)^{\frac{5}{2}} + \frac{2}{9}(x^3-1)^{\frac{3}{2}} + C\end{aligned}$$

2. Compute
$$\int x\ln(x^2+1)\,dx$$

   Solution:

   Let $u = x^2 + 1$, then $du = 2xdx$,

$$\begin{aligned}
\int x\ln(x^2+1)\,dx &= \frac{1}{2}\int \ln u\,du \\
&= \frac{1}{2}\int \ln u\,du \\
&= \frac{1}{2}(u\ln u - \int u\,d\ln u) \\
&= \frac{1}{2}(u\ln u - \int 1\,du) \\
&= \frac{1}{2}(u\ln u - u) + C \\
&= \frac{1}{2}(x^2+1)(\ln(x^2+1)-1) + C
\end{aligned}$$

3. Compute

$$\int_1^4 \frac{x-\sqrt{x}}{x+\sqrt{x}}\,dx$$

Solution: Let $x = t^2$, then

$$\begin{aligned}
\int_1^4 \frac{x-\sqrt{x}}{x+\sqrt{x}}\,dx = \int_1^2 \frac{t^2-t}{t^2+t}\,dt^2 &= \int_1^2 \frac{t(t-1)}{t(t+1)} \times 2t\,dt \\
&= \int_1^2 \frac{t-1}{t+1} \times 2t\,dt \\
&= \int_1^2 \frac{(t+1)-2}{t+1} \times 2t\,dt \\
&= \int_1^2 (1-\frac{2}{t+1}) \times 2t\,dt \\
&= \int_1^2 2t - \frac{4t}{t+1}\,dt \\
&= \int_1^2 2t - 4 + \frac{4}{t+1}\,dt \\
&= t^2 - 4t + 4\ln(t+1)\Big|_1^2 \\
&= -1 + 4\ln 3 - 4\ln 2
\end{aligned}$$

4. Compute
$$\int_1^e \frac{\ln x}{x}\,dx$$

Solution:

$$\begin{aligned}\int_1^e \frac{\ln x}{x}\,dx &= \int_1^e \ln x\,d\ln x\\ &= \frac{1}{2}(\ln x)^2\Big|_1^e\\ &= \frac{1}{2}\end{aligned}$$

5. Compute
$$\int_2^3 \frac{x}{x-1}\,dx$$

Solution:

$$\begin{aligned}\int_2^3 \frac{x}{x-1}\,dx &= \int_2^3 \frac{(x-1)+1}{x-1}\,d(x-1)\\ &= \int_2^3 1+\frac{1}{x-1}\,d(x-1)\\ &= (x-1)+\ln(x-1)\Big|_2^3\\ &= 1+\ln 2\end{aligned}$$

6. The demand function of some goods is $P = f(Q) = \frac{65}{Q+10}$, and the supply function is $P = g(Q) = Q + 2$. Compute the consumer surplus and producer surplus.

Solution:

We first find the equilibrium point (Q^*, P^*) by solving the equations

$$\begin{cases} P^* = \frac{65}{Q^*+10} \\ P^* = Q^* + 2 \end{cases}$$

We get $Q^* = 3,\ P^* = 5$.

The consumer surplus is

$$\int_0^{Q^*} f(Q)\,dQ - P^*Q^* = \int_0^3 \frac{65}{Q+10}\,dQ - 5\times 3 = 65\ln\frac{13}{10} - 15$$

The producer surplus is

$$P^*Q^* - \int_0^{Q^*} g(Q)\,dQ = 5\times 3 - \int_0^3 Q + 2\,dQ = \frac{9}{2}$$

2 Second-Half

1. In a forest, the proportion of trees shorter than x feet is given by

$$F(x) = \begin{cases} \frac{x}{10} - \frac{x^2}{400}, 0 \le x \le 20 \\ 1, x > 20 \end{cases}$$

   Compute the average height of the trees in the forest.

   Solution: $F(x) = 1$ for $x > 20$ implies that the height of any tree is no taller than 20, so we only need to study the interval $[0, 20]$. The density function is

$$f(x) = F'(x) = \frac{1}{10} - \frac{x}{200}$$

   The average height is

$$\frac{\int_0^{20} x f(x)\,dx}{\int_0^{20} f(x)\,dx} = \frac{\int_0^{20} x(\frac{1}{10} - \frac{x}{200})\,dx}{\int_0^{20} (\frac{1}{10} - \frac{x}{200})\,dx} = \frac{20}{3}$$

2. Calculate the double integral:

$$\iint_R ye^{-xy}\,dA$$

   where $R = [0, 2] \times [0, 3]$

   Solution:

$$
\begin{aligned}
\iint_R ye^{-xy}\,dA &= \int_0^3 \int_0^2 ye^{-xy}\,dx\,dy \\
&= \int_0^3 -e^{-xy}\Big|_{x=0}^{x=2}\,dy \\
&= \int_0^3 -e^{-2y} + 1\,dy \\
&= \frac{1}{2}e^{-2y} + y\Big|_{y=0}^{y=3} \\
&= \frac{e^{-6}+5}{2}
\end{aligned}
$$

3. Find the volume of the solid that lies under the plane $4x+6y-2z+15 = 0$ and above the rectangle $R = [-1,2] \times [1,2]$

   Solution: The surface can be rewritten as

$$
z = \frac{4x+6y+15}{2} = 2x+3y+\frac{15}{2}
$$

   The volume is

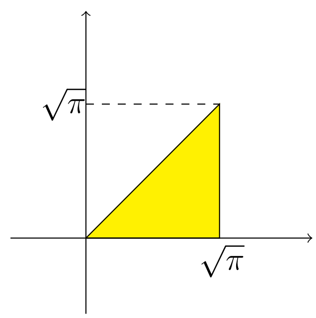
$$
\begin{aligned}
\int_{-1}^2 \int_1^2 2x+3y+\frac{15}{2}\,dy\,dx &= \int_{-1}^2 2xy+\frac{3}{2}y^2+\frac{15}{2}y\Big|_{y=1}^{y=2}\,dx \\
&= \int_{-1}^2 2x+12\,dx \\
&= 39
\end{aligned}
$$

4. Rewrite the following integral in the form of $\iint f(x,y)\,dy\,dx$

$$\int_0^{\sqrt{\pi}} \int_y^{\sqrt{\pi}} f(x,y)\,dx\,dy$$

Solution:

The region D is the following:



So the double integral can be written as

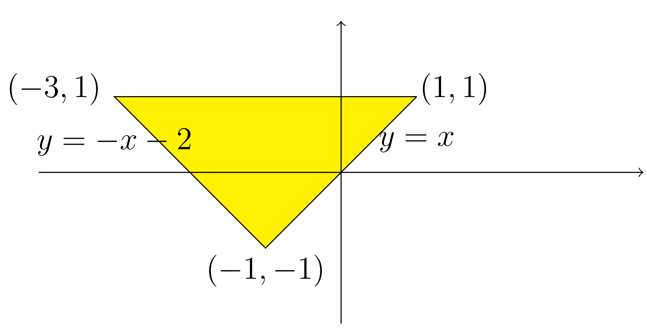
$$\int_0^{\sqrt{\pi}} \int_0^{x} f(x,y)\,dy\,dx$$

5. Evaluate the double integral

$$\iint_D y^2\,dA$$

where D is the triangle with vertexes $(1,1), (-1,-1), (-3,1)$

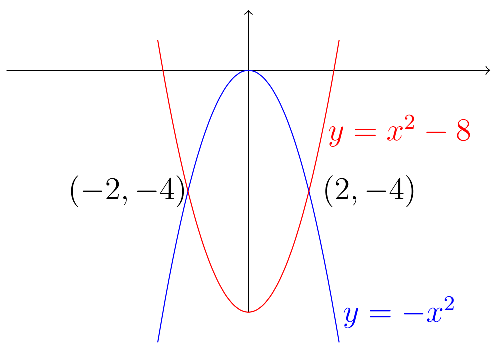
Solution:



$$
\begin{aligned}
\iint_D y^2\, dA &= \int_{-1}^{1} \int_{-y-2}^{y} y^2\, dx\, dy \\
&= \int_{-1}^{1} y^2 x \Big|_{x=-y-2}^{x=y} dy \\
&= \int_{-1}^{1} y^2(2y+2)\, dy \\
&= \frac{4}{3}
\end{aligned}
$$

6. Compute the area of the region bounded between the curves $y = -x^2$ and $y = x^2 - 8$ by double integral

   Solution:



$$
\begin{aligned}
\iint_D 1\, dA &= \int_{-2}^{2} \int_{x^2-8}^{-x^2} 1\, dy\, dx \\
&= \int_{-2}^{2} y \Big|_{x^2-8}^{y=-x^2} dx \\
&= \int_{-2}^{2} 8 - 2x^2\, dx \\
&= \frac{64}{3}
\end{aligned}
$$