Homework III

Solution

1 First-Half

1. Compute
$$\frac{d}{dx}\int_{2x+5}^{0} e^{8t}\,dt$$

   Solution:

$$\begin{aligned}\frac{d}{dx}\int_{2x+5}^{0} e^{8t}\,dt &= \frac{d}{dx}(-\int_{0}^{2x+5} e^{8t}\,dt)\\ &= -\frac{d}{dx}(\int_{0}^{2x+5} e^{8t}\,dt)\\ &= -e^{8(2x+5)}(2x+5)'\\ &= -2e^{16x+40}\end{aligned}$$

2. Compute the integral
$$\int_{1}^{2} \frac{x^3-1}{x}\,dx$$

   Solution:

$$\begin{aligned}\int_{1}^{2} \frac{x^3-1}{x}\,dx &= \int_{1}^{2} x^2 - \frac{1}{x}\,dx\\ &= \frac{x^3}{3} - \ln x\Big|_{1}^{2} = \frac{7}{3} - \ln 2\end{aligned}$$

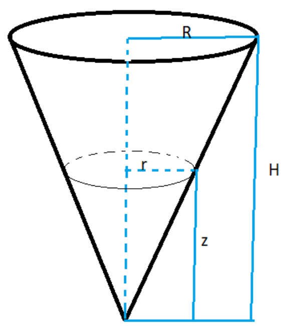
3. Prove the volume formula for a cone is

$$V = \frac{1}{3}\pi R^2 H$$

where R is the radius of the base circle and H is the height.

Solution:

We cut the cone into n horizontal pieces of equal thickness Δz. The radius of the top of each piece is $\frac{z_i}{H}R$ by similar triangles shown in the following picture.



So the sum of the volume of these n pieces can be approximated by

$$R_n = \sum_{i=1}^{n} \pi(\frac{z_i}{H}R)^2 \Delta z = \sum_{i=1}^{n} \frac{\pi R^2}{H^2} z^2 \Delta z$$

Taking the limit, we get the volume is

$$\lim_{n\to\infty} R_n = \int_0^H \frac{\pi R^2}{H^2} z^2\, dz = \frac{1}{3}\frac{\pi R^2}{H^2} z^3 \Big|_0^H = \frac{1}{3}\pi R^2 H$$

4. Compute the area of the region below $y = \ln x$ between $[1, e]$

Solution:

$$
\begin{aligned}
\int_1^e \ln x\,dx &= x\ln x\Big|_1^e - \int_1^e x\,d\ln x \\
&= e - \int_1^e x \times \frac{1}{x}\,dx \\
&= e - \int_1^e 1\,dx \\
&= e - (e-1) \\
&= 1
\end{aligned}
$$

5. Compute

$$\int xe^{-x}\,dx$$

Solution:

$$
\begin{aligned}
\int xe^{-x}\,dx &= -\int x\,de^{-x} \\
&= -xe^{-x} + \int e^{-x}\,dx \\
&= -xe^{-x} - e^{-x} + C
\end{aligned}
$$

6. Compute

$$\int_1^2 \sqrt{x}\ln x\,dx$$

Solution:

$$
\begin{aligned}
\int_1^2 \sqrt{x}\ln x\,dx = \frac{2}{3}(\int_1^2 \ln x\,dx^{\frac{3}{2}}) &= \frac{2}{3}(x^{\frac{3}{2}}\ln x\Big|_1^2 - \int_1^2 x^{\frac{3}{2}}\,d\ln x) \\
&= \frac{2}{3}(2\sqrt{2}\ln 2 - \int_1^2 x^{\frac{1}{2}}\,dx) \\
&= \frac{2}{3}(2\sqrt{2}\ln 2 - \frac{2}{3}x^{\frac{3}{2}}\Big|_1^2) \\
&= \frac{4}{3}\sqrt{2}\ln 2 - \frac{8}{9}\sqrt{2} + \frac{4}{9}
\end{aligned}
$$

2 Second-Half

1. Compute

$$\int_1^5 \ln 2x \, dx$$

Solution:

$$\begin{aligned}\int_1^5 \ln 2x \, dx &= x \ln 2x \Big|_1^5 - \int_1^5 x \, d \ln 2x \\ &= 5 \ln 10 - \ln 2 - \int_1^5 x \times \frac{2}{2x} \, dx \\ &= 5 \ln 10 - \ln 2 - \int_1^5 1 \, dx \\ &= 5 \ln 10 - \ln 2 - 4\end{aligned}$$

2. Compute

$$\int x^2 e^x \, dx$$

Solution:

$$\begin{aligned}\int x^2 e^x \, dx &= \int x^2 \, de^x \\ &= x^2 e^x - \int e^x \, dx^2 \\ &= x^2 e^x - 2 \int e^x x \, dx \\ &= x^2 e^x - 2 \int x \, de^x \\ &= x^2 e^x - 2(x e^x - \int e^x \, dx) \\ &= x^2 e^x - 2 x e^x + 2 e^x + C\end{aligned}$$

3. Compute

$$\int x^2 \ln x \, dx$$

Solution:

$$\begin{aligned}\int x^2 \ln x \, dx &= (x^2 \ln x)x - \int x \, d(x^2 \ln x) \\ &= x^3 \ln x - \int x(2x \ln x + x) \, dx \\ &= x^3 \ln x - 2\int x^2 \ln x \, dx - \int x^2 \, dx \\ &= x^3 \ln x - 2\int x^2 \ln x - \frac{1}{3}x^3\end{aligned}$$

So $3\int x^2 \ln x \, dx = x^3 \ln x - \frac{1}{3}x^3$, we conclude

$$\int x^2 \ln x \, dx = \frac{1}{3}x^3 \ln x - \frac{1}{9}x^3$$

4. Compute

$$\int e^{3x+1} \, dx$$

Solution:

$$\begin{aligned}\int e^{3x+1} \, dx &= \frac{1}{3}\int 1 \, de^{3x+1} \\ &= \frac{1}{3}(e^{3x+1} - \int e^{3x+1} 1' \, dx) \\ &= \frac{1}{3}e^{3x+1}\end{aligned}$$

5. Compute

$$\int_1^2 \frac{1}{x^2} \ln x \, dx$$

Solution:

$$\begin{aligned}
\int_1^2 \frac{1}{x^2}\ln x\,dx &= -\int_1^2 \ln x\,d\frac{1}{x} \\
&= -(\ln x)\frac{1}{x}\bigg|_1^2 + \int_1^2 \frac{1}{x}\,d\ln x \\
&= -\frac{\ln 2}{2} + \int_1^2 \frac{1}{x^2}\,dx \\
&= -\frac{\ln 2}{2} + \frac{1}{2}
\end{aligned}$$