Homework II

Solution

1 First-Half

1. The utility function is $u(x, y) = Ax^a y^b$. The unit price of $x$ is $p$ dollars and the unit price for $y$ is $q$ dollars. Find $x$ and $y$ to maximize the utility subject to $m$ dollars budget constraint.

Solution: The budget constraint function is

$$g(x, y) = px + qy = m$$

By the Lagrange multiplier method,

$$\begin{cases}
\frac{\partial u}{\partial x} = \lambda \frac{\partial g}{\partial x} \\
\frac{\partial u}{\partial y} = \lambda \frac{\partial g}{\partial y} \\
g(x, y) = m
\end{cases}$$

i.e.,

$$\begin{cases}
aAx^{a-1}y^b = \lambda p \\
bAx^a y^{b-1} = \lambda q \\
px + qy = m
\end{cases}$$

Taking the quotient of the first two equations, we see

$$\frac{ay}{bx} = \frac{p}{q}$$

Use this equation and $px + qy = m$, we can get $x = \frac{am}{(a+b)p}, \ y = \frac{bm}{(a+b)q}$
2. \( u(x, y) \) is a utility function, and \( g(x, y) \) is the budget function. When the budget is 200, it is known the maximal utility is 125. If we know at this point the Lagrange multiplier is \( \lambda = 3 \), estimate the maximal utility when the budget is 198.

**Solution:** Let \( u^*(m) \) denote the maximal utility when the budget constraint is \( m \). We know that

\[
u^*(198) - u^*(200) \approx \lambda (198 - 200) = 3 \times (-2) = -6
\]

So

\[
u^*(198) \approx u^*(200) - 6 = 125 - 6 = 119
\]

3. Minimize \( f(x, y, z) = x^2 + y^2 + z^2 \) subject to the constraint \( x + y + z = 1 \).

**Solution:**

By the method of Lagrange multiplier,

\[
\begin{align*}
\frac{\partial f}{\partial x} &= \lambda \frac{\partial g}{\partial x} \\
\frac{\partial f}{\partial y} &= \lambda \frac{\partial g}{\partial y} \\
\frac{\partial f}{\partial z} &= \lambda \frac{\partial g}{\partial z} \\
g(x, y, z) &= x + y + z = 1 
\end{align*}
\]

i.e.,

\[
\begin{align*}
2x &= \lambda \times 1 \\
2y &= \lambda \times 1 \\
2z &= \lambda \times 1 \\
x + y + z &= 1
\end{align*}
\]

Solving the system of equations, we get \( x = y = z = \frac{1}{3} \). So the minimum of \( f \) subject to \( x + y + z = 1 \) is \( f(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}) = \frac{1}{3} \).

4. Minimize \( f(x, y, z) = x^2 + y^2 + z^2 \) subject to the constraints \( x + y + z = 1 \) and \( x - y + z = 3 \)

**Solution:**

Let \( g(x, y, z) = x + y + z = 1 \), \( h(x, y, z) = x - y + z = 3 \).
By the method of Lagrange multiplier,
\[
\begin{aligned}
\frac{\partial f}{\partial x} &= \lambda \frac{\partial g}{\partial x} + \mu \frac{\partial h}{\partial x} \\
\frac{\partial f}{\partial y} &= \lambda \frac{\partial g}{\partial y} + \mu \frac{\partial h}{\partial y} \\
\frac{\partial f}{\partial z} &= \lambda \frac{\partial g}{\partial z} + \mu \frac{\partial h}{\partial z}
\end{aligned}
\]
\[
\begin{cases}
g(x, y, z) = x + y + z = 1 \\
h(x, y, z) = x - y + z = 3
\end{cases}
\]
i.e.,
\[
\begin{aligned}
2x &= \lambda \times 1 + \mu \times 1 \\
2y &= \lambda \times 1 + \mu \times (-1) \\
2z &= \lambda \times 1 + \mu \times 1 \\
x + y + z &= 1 \\
x - y + z &= 3
\end{aligned}
\]
Solving the above system of equations, we get \(x = 1, y = -1, z = 1\).
So the minimum of \(f\) subject to those constraints is \(f(1, -1, 1) = 3\).

5. There is a string of length \(L\). Mickey cuts the string into two segments, of length \(x\) and \(y\) respectively. He uses the length \(x\) segment to make a square and the length \(y\) segment to make a circle. Find \((x, y)\) that minimizes the total area enclosed by the square and the circle.

\[
\begin{aligned}
&\text{Solution:} \\
&\text{The area of the square is } (\frac{x}{4})^2 = \frac{x^2}{16}, \text{ and the area of the circle is } \\
&\pi(\frac{y}{2\pi})^2 = \frac{y^2}{4\pi}, \text{ so the total area is} \\
&f(x, y) = \frac{x^2}{16} + \frac{y^2}{4\pi}
\end{aligned}
\]
We are going to minimize \(f\) subject to the constraint \(g(x, y) = x + y = L\).
\[
\begin{align*}
\frac{\partial f}{\partial x} &= \lambda \frac{\partial g}{\partial x} \\
\frac{\partial f}{\partial y} &= \lambda \frac{\partial g}{\partial y} \\
g(x, y) &= L
\end{align*}
\]

i.e.,
\[
\begin{align*}
\frac{\partial f}{\partial x} &= \lambda \times 1 \\
\frac{\partial f}{\partial y} &= \lambda \times 1 \\
x + y &= L
\end{align*}
\]

We get \(x = \frac{4L}{4+\pi}\) and \(y = \frac{\pi L}{4+\pi}\)

## 2 Second-Half

1. Find the most general form of the antiderivatives of

\[f(x) = 5x^4 + 3e^{2x} - \frac{2}{x} - \frac{1}{x^2}\]

on the interval \((0, +\infty)\)

**Solution:**

\[F(x) = x^5 + \frac{3}{2}e^{2x} - 2\ln x + \frac{1}{x} + C\]

2. \(f''(x) = 2x^3 - 5x, \ f'(1) = 2, \ f(1) = 4. \) What is \(f(x)\)?

**Solution:**

\[
\begin{align*}
f'(x) &= \frac{1}{2}x^4 - \frac{5}{2}x^2 + C_1 \\
f(x) &= \frac{1}{10}x^5 - \frac{5}{6}x^3 + C_1x + C_2
\end{align*}
\]

\[
\begin{align*}
2 &= f'(1) = \frac{1}{2} - \frac{5}{2} + C_1 = -2 + C_1 \\
4 &= f(1) = \frac{1}{10} - \frac{5}{6} + C_1 + C_2 = -\frac{11}{15} + C_1 + C_2
\end{align*}
\]

We get \(C_1 = 4\) and \(C_2 = \frac{11}{15}\)

So \(f(x) = \frac{1}{10}x^5 - \frac{5}{6}x^3 + 4x + \frac{11}{15}\)

3. Use \(R_n\) to compute the area of the region under the curve \(y = x + 1\) on \([1, 2]\). (You may need to use \(\sum_{i=1}^{n} i = \frac{n(n+1)}{2}\))

**Solution:**
Let $\Delta x = \frac{2 - 1}{n} = \frac{1}{n}$.
$x_0 = 1, x_1 = 1 + \frac{1}{n}, ..., x_i = 1 + \frac{i}{n}, ..., x_n = 1 + \frac{n}{n} = 2$

\[
R_n = \sum_{i=1}^{n} f(x_i) \Delta x \\
= \sum_{i=1}^{n} (2 + \frac{i}{n}) \frac{1}{n} \\
= \sum_{i=1}^{n} \left(\frac{2n + i}{n}\right) \frac{1}{n} \\
= \frac{1}{n^2} \left(\sum_{i=1}^{n} 2n + i\right) \\
= \frac{1}{n^2} \left(\sum_{i=1}^{n} 2n + \sum_{i=1}^{n} i\right) \\
= \frac{1}{n^2} \left(2n^2 + \frac{n(n+1)}{2}\right) \\
= 2 + \frac{n+1}{2n} \\
= \frac{5}{2} + \frac{1}{2n}
\]

So the area is $\lim_{n \to \infty} R_n = \frac{5}{2}$

4. $f(x) = |x| - \sqrt{1 - x^2}$. What is $\int_{-1}^{1} f(x) \, dx$?

Solution:

$\int_{-1}^{1} f(x) \, dx = \int_{-1}^{1} |x| \, dx - \int_{-1}^{1} \sqrt{1 - x^2} \, dx$. By studying the graphs of $y = |x|$ and $y = \sqrt{1 - x^2}$, the integrals can be obtained by computing the area bounded between the graphs and $x$-axis on $[-1, 1]$, see figure 4 so we conclude

\[
\int_{-1}^{1} f(x) \, dx = \int_{-1}^{1} |x| \, dx - \int_{-1}^{1} \sqrt{1 - x^2} \, dx = 2 \times \frac{1}{2} - \frac{1}{2} \times \pi \times 1^2 = 1 - \frac{\pi}{2}
\]
5. Use the Midpoint rule when \( n = 6 \) to estimate \( \int_{0}^{3} x^2 \, dx \).

**Solution:**

\[ \Delta x = \frac{3-0}{6} = 0.5, \text{ so } x_0 = 0, x_1 = 0.5, x_2 = 1, x_3 = 1.5, x_4 = 2, x_5 = 2.5, x_6 = 3 \]

We get \( x^*_1 = 0.25, x^*_2 = 0.75, x^*_3 = 1.25, x^*_4 = 1.75, x^*_5 = 2.25, x^*_6 = 2.75 \), so

\[
\int_{0}^{3} x^2 \, dx \approx \sum_{n=1}^{6} f(x^*_n) \Delta x \\
= (0.25^2 + 0.75^2 + 1.25^2 + 1.75^2 + 2.25^2 + 2.75^2) \times 0.5 \\
= 8.9375
\]