Homework II

Solution

1 First-Half

1. The utility function is $u(x,y) = Ax^a y^b$. The unit price of x is p dollars and the unit price for y is q dollars. Find x and y to maximize the utility subject to m dollars budget constraint.

   Solution: The budget constraint function is

   $$g(x,y) = px + qy = m$$

   By the Lagrange multiplier method,

   $$\begin{cases} \frac{\partial u}{\partial x} = \lambda \frac{\partial g}{\partial x} \\ \frac{\partial u}{\partial y} = \lambda \frac{\partial g}{\partial y} \\ g(x,y) = m \end{cases}$$

   i.e.,

   $$\begin{cases} aAx^{a-1}y^b = \lambda p \\ bAx^a y^{b-1} = \lambda q \\ px + qy = m \end{cases}$$

   Taking the quotient of the first two equations, we see

   $$\frac{ay}{bx} = \frac{p}{q}$$

   Use this equation and $px + qy = m$, we can get $x = \frac{am}{(a+b)p}$, $y = \frac{bm}{(a+b)q}$

2. $u(x,y)$ is a utility function, and $g(x,y)$ is the budget function. When the budget is 200, it is known the maximal utility is 125. If we know at this point the Lagrange multiplier is $\lambda = 3$, estimate the maximal utility when the budget is 198

   Solution: Let $u^*(m)$ denote the maximal utility when the budget constraint is m. We know that

   $$u^*(198) - u^*(200) \approx \lambda(198 - 200) = 3 \times (-2) = -6$$

   So

   $$u^*(198) \approx u^*(200) - 6 = 125 - 6 = 119$$

3. Minimize $f(x,y,z) = x^2+y^2+z^2$ subject to the constraint $x+y+z = 1$.

   Solution:

   By the method of Lagrange multiplier,

   $$\begin{cases} \frac{\partial f}{\partial x} = \lambda \frac{\partial g}{\partial x} \\ \frac{\partial f}{\partial y} = \lambda \frac{\partial g}{\partial y} \\ \frac{\partial f}{\partial z} = \lambda \frac{\partial g}{\partial z} \\ g(x,y,z) = x+y+z = 1 \end{cases}$$

   i.e.,

   $$\begin{cases} 2x = \lambda \times 1 \\ 2y = \lambda \times 1 \\ 2z = \lambda \times 1 \\ x+y+z = 1 \end{cases}$$

   Solving the system of equations, we get $x = y = z = \frac{1}{3}$. So the minimum of f subject to $x+y+z = 1$ is $f(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}) = \frac{1}{3}$

4. Minimize $f(x,y,z) = x^2+y^2+z^2$ subject to the constraints $x+y+z = 1$ and $x-y+z = 3$

   Solution:

   Let $g(x,y,z) = x+y+z = 1$, $h(x,y,z) = x-y+z = 3$.

By the method of Lagrange multiplier,

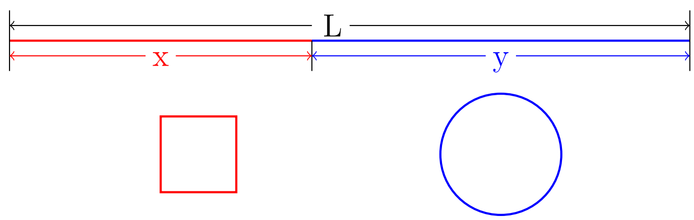
$$\begin{cases} \frac{\partial f}{\partial x} = \lambda \frac{\partial g}{\partial x} + \mu \frac{\partial h}{\partial x} \\ \frac{\partial f}{\partial y} = \lambda \frac{\partial g}{\partial y} + \mu \frac{\partial h}{\partial y} \\ \frac{\partial f}{\partial z} = \lambda \frac{\partial g}{\partial z} + \mu \frac{\partial h}{\partial z} \\ g(x,y,z) = x + y + z = 1 \\ h(x,y,z) = x - y + z = 3 \end{cases}$$

i.e.,

$$\begin{cases} 2x = \lambda \times 1 + \mu \times 1 \\ 2y = \lambda \times 1 + \mu \times (-1) \\ 2z = \lambda \times 1 + \mu \times 1 \\ x + y + z = 1 \\ x - y + z = 3 \end{cases}$$

Solving the above system of equations, we get $x = 1, y = -1, z = 1$. So the minimum of f subject to those constraints is $f(1,-1,1) = 3$

5. There is a string of length L. Mickey cuts the string into two segments, of length x and y respectively. He uses the length x segment to make a square and the length y segment to make a circle. Find (x, y) that minimizes the total area enclosed by the square and the circle.



Solution:

The area of the square is $(\frac{x}{4})^2 = \frac{x^2}{16}$, and the area of the circle is $\pi(\frac{y}{2\pi})^2 = \frac{y^2}{4\pi}$, so the total area is

$$f(x,y) = \frac{x^2}{16} + \frac{y^2}{4\pi}$$

We are going to minimize f subject to the constraint $g(x,y) = x + y = L$.

$$\begin{cases} \frac{\partial f}{\partial x} = \lambda \frac{\partial g}{\partial x} \\ \frac{\partial f}{\partial y} = \lambda \frac{\partial g}{\partial y} \\ g(x, y) = L \end{cases}$$

i.e.,

$$\begin{cases} \frac{x}{8} = \lambda \times 1 \\ \frac{y}{2\pi} = \lambda \times 1 \\ x + y = L \end{cases}$$

We get $x = \frac{4L}{4+\pi}$ and $y = \frac{\pi L}{4+\pi}$

2 Second-Half

1. Find the most general form of the antiderivatives of

$$f(x) = 5x^4 + 3e^{2x} - \frac{2}{x} - \frac{1}{x^2}$$

on the interval $(0, +\infty)$

Solution:

$F(x) = x^5 + \frac{3}{2}e^{2x} - 2\ln x + \frac{1}{x} + C$

2. $f''(x) = 2x^3 - 5x,\ f'(1) = 2,\ f(1) = 4$. What is $f(x)$?

Solution:

$f'(x) = \frac{1}{2}x^4 - \frac{5}{2}x^2 + C_1$

$f(x) = \frac{1}{10}x^5 - \frac{5}{6}x^3 + C_1x + C_2$

$$\begin{cases} 2 = f'(1) = \frac{1}{2} - \frac{5}{2} + C_1 = -2 + C_1 \\ 4 = f(1) = \frac{1}{10} - \frac{5}{6} + C_1 + C_2 = -\frac{11}{15} + C_1 + C_2 \end{cases}$$

We get $C_1 = 4$ and $C_2 = \frac{11}{15}$

So $f(x) = \frac{1}{10}x^5 - \frac{5}{6}x^3 + 4x + \frac{11}{15}$

3. Use R_n to compute the area of the region under the curve $y = x + 1$ on $[1, 2]$. (You may need to use $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$)

Solution:

Let $\Delta x = \frac{2-1}{n} = \frac{1}{n}$.
$x_0 = 1, x_1 = 1 + \frac{1}{n}, ..., x_i = 1 + \frac{i}{n}, ..., x_n = 1 + \frac{n}{n} = 2$

$$\begin{aligned}
R_n &= \sum_{i=1}^{n} f(x_i)\Delta x \\
&= \sum_{i=1}^{n} (2 + \frac{i}{n})\frac{1}{n} \\
&= \sum_{i=1}^{n} (\frac{2n+i}{n})\frac{1}{n} \\
&= \frac{1}{n^2}(\sum_{i=1}^{n} 2n + i) \\
&= \frac{1}{n^2}(\sum_{i=1}^{n} 2n + \sum_{i=1}^{n} i) \\
&= \frac{1}{n^2}(2n^2 + \frac{n(n+1)}{2}) \\
&= 2 + \frac{n+1}{2n} \\
&= \frac{5}{2} + \frac{1}{2n}
\end{aligned}$$

So the area is $\lim_{n\to\infty} R_n = \frac{5}{2}$

4. $f(x) = |x| - \sqrt{1-x^2}$. What is $\int_{-1}^{1} f(x)\,dx$?

Solution:
$\int_{-1}^{1} f(x)\,dx = \int_{-1}^{1} |x|\,dx - \int_{-1}^{1} \sqrt{1-x^2}\,dx$. By studying the graphs of $y = |x|$ and $y = \sqrt{1-x^2}$, the integrals can be obtained by computing the area bounded between the graphs and x-axis on $[-1, 1]$, see figure 4 so we conclude

$$\int_{-1}^{1} f(x)\,dx = \int_{-1}^{1} |x|\,dx - \int_{-1}^{1} \sqrt{1-x^2}\,dx = 2\times\frac{1\times 1}{2} - \frac{1}{2}\times\pi\times 1^2 = 1 - \frac{\pi}{2}$$

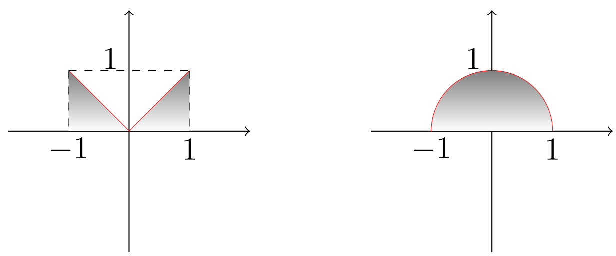


Figure 1:

5. Use the Midpoint rule when $n = 6$ to estimate $\int_0^3 x^2\, dx$.

   Solution:

   $\Delta x = \frac{3-0}{6} = 0.5$, so $x_0 = 0, x_1 = 0.5, x_2 = 1, x_3 = 1.5, x_4 = 2, x_5 = 2.5, x_6 = 3$

   We get $x_1^* = 0.25, x_2^* = 0.75, x_3^* = 1.25, x_4^* = 1.75, x_5^* = 2.25, x_6^* = 2.75$, so

$$\begin{aligned}\int_0^3 x^2\, dx &\approx \sum_{n=1}^{6} f(x_i^*)\Delta x \\ &= (0.25^2 + 0.75^2 + 1.25^2 + 1.75^2 + 2.25^2 + 2.75^2) \times 0.5 \\ &= 8.9375\end{aligned}$$