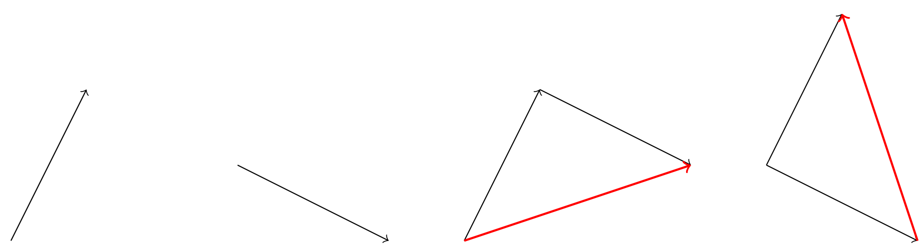
Homework I

Solution

1 First-Half

1. The vectors $\vec{u}$ and $\vec{v}$ are given as below. Sketch $\vec{u}+\vec{v}$ and $\vec{u}-\vec{v}$



$\vec{u}$ $\quad$ $\vec{v}$ $\quad$ $\vec{u}+\vec{v}$ $\quad$ $\vec{u}-\vec{v}$

2. If $\vec{u}=(1,3)$ and $\vec{v}=(-2,5)$, compute $\vec{u}+\vec{v}$, $\vec{u}-\vec{v}$, $-3\vec{u}$ and $\vec{u}.\vec{v}$

   Solution:

   $\vec{u}+\vec{v}=(1,3)+(-2,5)=(1+(-2),3+5)=(-1,8)$.

   $\vec{u}-\vec{v}=(1,3)-(-2,5)=(1-(-2),3-5)=(3,-2)$

   $-3\vec{u}=-3(1,3)=(-3\times 1,-3\times 3)=(-3,-9)$

   $\vec{u}\cdot\vec{v}=(1,3)\cdot(-2,5)=1\times(-2)+3\times 5=13$

3. Find a unit vector that has the opposite direction as $\vec{v}=(5,-12)$

   Solution:

$|\vec{v}| = \sqrt{5^2 + (-12)^2} = 13$. The vector that has opposite direction as $\vec{v}$ is thus

$$-\frac{1}{|\vec{v}|}\vec{v} = -\frac{1}{13}(5, -12) = (-\frac{5}{13}, \frac{12}{13})$$

4. $\vec{u}$ and $\vec{v}$ are vectors such that $(\vec{u} + \vec{v}) \perp (\vec{u} - \vec{v})$, prove $|\vec{u}| = |\vec{v}|$.

   Solution: There is more than one way to do it.

   Method I:

   If $\vec{u} + \vec{v} \perp \vec{u} - \vec{v}$, then

$$\begin{aligned}
(\vec{u} + \vec{v}) \cdot (\vec{u} - \vec{v}) &= 0 \\
\vec{u} \cdot \vec{u} - \vec{u} \cdot \vec{v} + \vec{v} \cdot \vec{u} - \vec{v} \cdot \vec{v} &= 0 \\
\vec{u} \cdot \vec{u} - \vec{v} \cdot \vec{v} &= 0 \\
|\vec{u}|^2 - |\vec{v}|^2 &= 0 \\
|\vec{u}|^2 &= |\vec{v}|^2 \\
|\vec{u}| &= |\vec{v}|
\end{aligned}$$

   Method II:

   Assume $\vec{u} = (x_1, y_1)$, $\vec{v} = (x_2, y_2)$.

$$\begin{aligned}
(\vec{u} + \vec{v}) \cdot (\vec{u} - \vec{v}) &= 0 \\
(x_1 + x_2, y_1 + y_2) \cdot (x_1 - x_2, y_1 - y_2) &= 0 \\
(x_1 + x_2)(x_1 - x_2) + (y_1 + y_2)(y_1 - y_2) &= 0 \\
x_1^2 - x_2^2 + y_1^2 - y_2^2 &= 0 \\
x_1^2 + y_1^2 &= x_2^2 + y_2^2 \\
|\vec{u}|^2 &= |\vec{v}|^2 \\
|\vec{u}| &= |\vec{v}|
\end{aligned}$$

5. $f(x, y) = x^2 + xy + 2y^2$. $\vec{u} = (\frac{3}{5}, -\frac{4}{5})$ is a unit vector.

   (i). Compute the gradient ∇f

   (ii). Compute the directional derivative $D_{\vec{u}} f(1, 3)$.

   Solution:

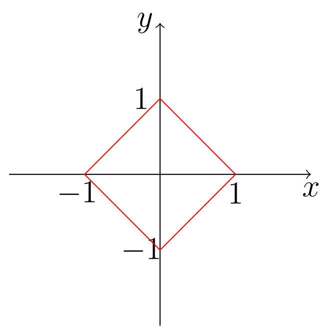
(i). $\nabla f = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}) = (2x + y, x + 4y)$

(ii). $\nabla f(1,3) = (5,13)$, so $D_{\vec{u}} f(1,3) = \nabla f(1,3).\vec{u} = -\frac{37}{5}$

2 Second-Half

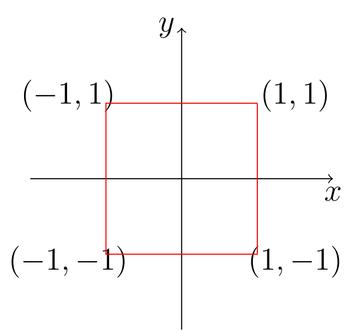
1. $f(x,y) = |x| + |y|$. Draw the level set $f(x,y) = 1$.

   Solution:



2. $f(x,y) = \max\{|x|,|y|\}$. Draw the level set $f(x,y) = 1$.

   Solution:



3. Suppose you are climbing a hill whose shape is given by the equation $z = 1000 - 0.005x^2 - 0.01y^2$, where x, y, z are measured in meters. You are standing at a point with coordinates $(60, 40, 966)$. The positive x-axis points east and the positive y-axis points north.

   (a). If you walk due south, will you start to ascend or descend? At what rate?

(b). If you walk northwest, will you start to ascend or descend? At what rate?

(c). In which direction is the slope largest? What is the rate of ascend in that direction? At what angle above the horizon does the path in that direction begin?

Solution:

The gradient vector is $\nabla z = (\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}) = (-0.01x, -0.02y)$, at $(60, 40, 966)$ it is $\nabla z = (-0.6, -0.8)$

(a). The south direction has unit directional vector $\vec{u} = (0, -1)$, so

$$D_{\vec{u}} z = (-0.6, -0.8) \cdot (0, -1) = 0.8$$

So you will start to ascend at rate of 0.8

(b). The northwest direction has unit directional vector $\vec{u} = (-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$, so

$$D_{\vec{u}} z = (-0.6, -0.8) \cdot (-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}) = -\frac{\sqrt{2}}{10}$$

So you will start to descend at rate of $\frac{\sqrt{2}}{10}$

(c). The slope is largest in the direction of the gradient $\nabla z = (-0.6, -0.8)$. The rate of ascend is $|(-0.6, -0.8)| = 1$. If we assume the angle is θ, then $\tan\theta = 1$, we get $\theta = \frac{\pi}{4}$.

4. Maximize the function $f(x, y) = 2x - y$ under the constraint $x^2 + y^2 = 4$

Solution:

Let $g(x, y) = x^2 + y^2 = 4$.

$\nabla f(x, y) = (2, -1)$, and $\nabla g(x, y) = (2x, 2y)$

We have

$$\begin{cases} \nabla f(x, y) = \lambda \nabla g(x, y) \\ g(x, y) = 4 \end{cases}$$

i.e.

$$\begin{cases} 2 = \lambda 2x \\ -1 = \lambda 2y \\ x^2 + y^2 = 4 \end{cases}$$

Solutions are $\lambda = \frac{\sqrt{5}}{4}, x = \frac{4}{\sqrt{5}}, y = -\frac{2}{\sqrt{5}}$ or $\lambda = -\frac{\sqrt{5}}{4}, x = -\frac{4}{\sqrt{5}}, y = \frac{2}{\sqrt{5}}$. $f(\frac{4}{\sqrt{5}}, -\frac{2}{\sqrt{5}}) = 2\sqrt{5}$ and $f(-\frac{4}{\sqrt{5}}, \frac{2}{\sqrt{5}}) = -2\sqrt{5}$, we conclude the maximum value is $2\sqrt{5}$, obtained when $x = \frac{4}{\sqrt{5}}, y = -\frac{2}{\sqrt{5}}$

5. Use the Lagrange multiplier method to find the largest possible area of a rectangle all of whose vertexes are on a circle of radius 2.

   Solution:

   Let the length and width of the edges of the rectangle be x and y respectively. The area is $f(x, y) = xy$. Since the vertexes are on a circle of radius 2, by Pythagorean Theorem, we have the constraint $g(x, y) = (\frac{x}{2})^2 + (\frac{y}{2})^2 = 2^2$, i.e. $g(x, y) = \frac{x^2}{4} + \frac{y^2}{4} = 4$

   Applying the method of Lagrange multiplier, we have

$$\begin{cases} \nabla f(x, y) = \lambda \nabla g(x, y) \\ g(x, y) = 4 \end{cases}$$

   i.e.

$$\begin{cases} y = \lambda \frac{x}{2} \\ x = \lambda \frac{y}{2} \\ \frac{x^2}{4} + \frac{y^2}{4} = 4 \end{cases}$$

   Also note that we need $x > 0, y > 0$, so the only solution is $\lambda = 2, x = 2\sqrt{2}, y = 2\sqrt{2}$, and the maximal area is $f(2\sqrt{2}, 2\sqrt{2}) = 8$.