Integration by Parts

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We have seen that by the Fundamental Theorem of Calculus, we can evaluate a given integral if we can find an antiderivative of the integrand function. But in general, it is not easy to find antiderivatives directly, so we need more techniques.

**Theorem 1.** (*Integration by Parts*)

\[
\int f(x)g'(x) = f(x)g(x) - \int g(x)f'(x) \, dx
\]

**Proof.** By the Leibniz Rule of differentiating a product of functions, we know

\[
(f(x)g(x))' = f'(x)g(x) + f(x)g'(x)
\]

So \( f(x)g(x) \) is an antiderivative of \( f'(x)g(x) + f(x)g'(x) \),

\[
\int f'(x)g(x) + f(x)g'(x) \, dx = f(x)g(x) + C
\]

We then see

\[
\int f(x)g'(x) \, dx = f(x)g(x) - \int g(x)f'(x) \, dx
\]

\[\square\]

**Remark 2.** Recall that given a differentiable function \( f \), there is a corresponding differential \( df = f'(x)dx \), so the above theorem can also be written as

\[
\int f(x) \, dg(x) = f(x)g(x) - \int g(x) \, df(x)
\]

We can use the above theorem to find antiderivatives of product of functions.
Example 3. Find antiderivatives of $f(x) = xe^x$.

$$\int xe^x \, dx = \int x(e^x)' \, dx$$
$$= xe^x - \int (x)'e^x \, dx$$
$$= xe^x - \int e^x \, dx$$
$$= xe^x - e^x + C$$

If you like to use the differential notation instead, you will get

$$\int xe^x \, dx = \int x \, de^x$$
$$= xe^x - \int e^x \, dx$$
$$= xe^x - e^x + C$$

Example 4. Compute $\int \frac{1}{x} \ln x \, dx$

$$\int \frac{1}{x} \ln x \, dx = \int (\ln x)' \ln x \, dx$$
$$= (\ln x)(\ln x) - \int \ln x (\ln x)' \, dx$$
$$= (\ln x)^2 - \int \frac{1}{x} \ln x \, dx$$

We see $\int \frac{1}{x} \ln x \, dx = \frac{1}{2}(\ln x)^2 + C$

We can also apply the method of Integration by Parts in evaluating definite integrals:

Theorem 5.

$$\int_a^b f(x)g'(x) \, dx = f(b)g(b) - f(a)g(a) - \int_a^b g(x)f'(x) \, dx$$
Proof. We have seen that \( f(x)g(x) \) is an antiderivative of \( f(x)g'(x) + f'(x)g(x) \). Then by the Fundamental Theorem of Calculus,

\[
\int_a^b f(x)g'(x) + f'(x)g(x) \, dx = f(b)g(b) - f(a)g(a)
\]

\[\square\]

Example 6. Evaluate \( \int_1^3 x \ln x \, dx \)

\[
\int_1^3 x \ln x \, dx = \int_1^3 \ln x \frac{x^2}{2} \, dx
\]

\[
= \frac{x^2}{2} \ln x \bigg|_1^3 - \int_1^3 \frac{x^2}{2} \, d\ln x
\]

\[
= \frac{9}{2} \ln 3 - \int_1^3 \frac{x^2}{2} \, dx
\]

\[
= \frac{9}{2} \ln 3 - \left[ \frac{x^3}{6} \right]_1^3
\]

\[
= \frac{9}{2} \ln 3 - 2
\]