Double Integrals (Part II)

Liming Pang

1 Double Integral on General Regions

In many cases, we want to compute the volume of some solid whose base is not a rectangle. In such situation, we need to define and compute the double integrals over a general region. The idea is to extend the given function to a larger rectangular domain by assigning the value 0 to the points out of \( D \):

We define \( \iint_D f(x,y) \, dA = \iint_R \tilde{f}(x,y) \, dA \), where \( R \) is a rectangular region enclosing \( D \) and

\[
\tilde{f}(x,y) = \begin{cases} 
  f(x,y), & \text{if } (x,y) \text{ is in } D \\
  0, & \text{if } (x,y) \text{ is not in } D 
\end{cases}
\]

**Proposition 1.** \( D \) is a region on the \( xy \)-plane, then:

1. \( \iint_D f(x,y) + g(x,y) \, dA = \iint_D f(x,y) \, dA + \iint_D g(x,y) \, dA \)
2. \( \iint_D cf(x,y) \, dA = c \iint_D f(x,y) \, dA \)
3. If \( f(x,y) \geq g(x,y) \) on \( D \), then \( \iint_D f(x,y) \, dA \geq \iint_D g(x,y) \, dA \)

A region \( D \) is said to be of Type I if it is bounded by \( x = a, \ x = b, \ y = g_1(x) \) and \( y = g_2(x) \), where \( g_1, g_2 \) are continuous functions in \( x \) and \( g_1(x) \leq g_2(x) \) on \([a,b]\).

The double integral over a Type I region can be computed as follows:

**Theorem 2.**

\[
\iint_D f(x,y) \, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y) \, dy \, dx
\]
Proof. We use a rectangle $R = [a, b] \times [c, d]$ to enclose this region $D$.

\[
\iint_D f(x, y) \, dA = \iint_R \tilde{f}(x, y) \, dA = \int_a^b \int_c^d \tilde{f}(x, y) \, dy \, dx = \int_a^b \left( \int_c^{g_1(x)} \tilde{f}(x, y) \, dy + \int_{g_1(x)}^{g_2(x)} \tilde{f}(x, y) \, dy + \int_{g_2(x)}^d \tilde{f}(x, y) \, dy \right) \, dx
\]

Similarly, we can define a plane region to be of Type II if it is bounded by $y = c, y = d, x = h_1(y)$ and $x = h_2(y)$, where $h_1, h_2$ are continuous functions on $[c, d]$ and $h_1(y) \leq h_2(y)$ on $[c, d]$.

Then the integral on $D$ is given by

**Theorem 3.**

\[
\iint_D f(x, y) \, dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y) \, dy \, dx
\]
Example 4. Compute the integral $\int \int_D xy \, dA$ over the triangular shaded region shown in the following figure.

\[ \int \int_D f(x,y) \, dA = \int_0^1 \int_0^x xy \, dy \, dx = \int_0^1 \left( \int_0^x \frac{x^3}{2} \, dy \right) \, dx \]
\[ = \int_0^1 \frac{x^3}{2} \, dx \]
\[ = \frac{1}{8} \]
Method II:

\[
\iint_D f(x, y) \, dA = \int_0^1 \int_y^1 xy \, dx \, dy
\]
\[
= \int_0^1 \left[ \frac{y}{2} x^2 \right]_y^1 \, dy
\]
\[
= \int_0^1 \frac{y - y^3}{2} \, dy
\]
\[
= \frac{1}{8}
\]

Example 5. Find the volume of the solid that lies under the graph \( z = f(x, y) = x^2 + y^2 \) and above the region \( D \) in \( xy \)-plane bounded by the line \( y = 2x \) and the parabola \( y = x^2 \).

\[
\iint_D f(x, y) \, dA = \int_0^2 \int_0^{\sqrt{y}} x^2 + y^2 \, dx \, dy
\]
\[
= \int_0^2 \left( \frac{x^3}{3} + \frac{14}{3} x^3 \right) \, dx
\]
\[
= \frac{216}{35}
\]

Example 6. Rewrite the integral in the above example in the form of \( \iint f(x, y) \, dx \, dy \)

\[
\iint_D f(x, y) \, dA = \int_0^4 \int_{\frac{y}{2}}^\sqrt{y} x^2 + y^2 \, dx \, dy
\]
Example 7. Rewrite the iterated integral \( \int_{-3}^{0} \int_{0}^{y^2} f(x, y) \, dx \, dy \) in the form of \( \int \int f(x, y) \, dy \, dx \)

By the given iterated integral, we can recover the region \( D \) to be the following:

So the integral can be written as

\[
\int_{0}^{9} \int_{-3}^{-\sqrt{x}} f(x, y) \, dy \, dx
\]

Proposition 8. Given a region \( D \) on xy-plane, its area is

\[
\int \int_{D} 1 \, dA
\]

Example 9. Compute the area bounded between the curve \( x = y^2 \) and \( x = 4 \)

\[
\int \int_{D} 1 \, dA = \int_{0}^{4} \int_{-\sqrt{x}}^{\sqrt{x}} 1 \, dy \, dx = \int_{0}^{4} 2\sqrt{x} \, dx = \frac{32}{3}
\]

An important application of double integral is to compute the mass of some thin object.

Proposition 10. If a thin object is put on the xy-plane, it occupies a region \( D \). If the density function of this object is \( \rho(x, y) \), then its mass is given by

\[
\int \int_{D} \rho(x, y) \, dA
\]