Directional Derivative, Gradient and Level Set

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1 Directional Derivative

The partial derivatives of a multi-variable function $f(x,y)$, $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$, tell us the rate of change of the function along the x-axis and y-axis respectively. But in general what about the rate of change in other directions?

On the xy-plane, each direction can be represented by a unit vector $\vec{u}$. We are going to define the directional derivative of a function $z = f(x,y)$ at (x_0, y_0) in the direction $\vec{u}$:

On the xy-plane, consider the line l passing through (x_0, y_0) and parallel to the unit vector $\vec{u}$. Passing through the line l, there is a unique vertical plane α, and α intersects the graph of $z = f(x,y)$ along a curve C, so C projects to l on the xy-plane.

If we start at (x_0, y_0) and travel along $\vec{u}$ direction for a distance h, arriving at (x,y). Then the vector with initial point (x_0, y_0) and terminal point (x,y) is $(x - x_0, y - y_0) = h\vec{u}$. Since $\vec{u}$ is a unit vector, let $\vec{u} = (a,b)$, where $a^2 + b^2 = 1$. So:

$$(x - x_0, y - y_0) = h\vec{u} = (ha, hb)$$

which implies

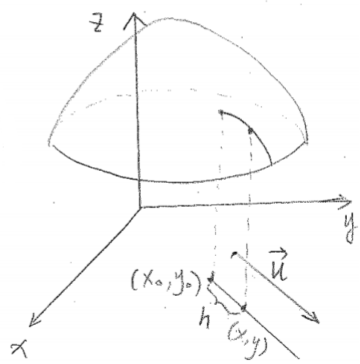
$$x = x_0 + ha, y = y_0 + hb$$

so the rate of change of f along $\vec{u}$ at (x_0, y_0) is

$$D_{\vec{u}} f(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}$$

We call it the directional derivative of f at (x_0, y_0) in the direction of $\vec{u}$.

Indeed there is a faster way to evaluate the directional derivative.



Theorem 1. *If f is a differentiable function of x and y, then f has a directional derivative in the direction of any unit vector $\vec{u} = (a, b)$ and*

$$D_{\vec{u}} f(x_0, y_0) = a\frac{\partial f}{\partial x}(x_0, y_0) + b\frac{\partial f}{\partial y}(x_0, y_0)$$

Proof. Define $g(h) = f(x_0 + ha, y_0 + hb)$. We get

$$g'(0) = \lim_{h\to 0} \frac{g(h) - g(0)}{h} = \lim_{h\to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h} = D_{\vec{u}} f(x_0, y_0)$$

On the other hand, by the chain rule,

$$\begin{aligned} g'(h) &= \frac{\partial f}{\partial x}(x_0 + ha, y_0 + hb)\frac{d(x_0 + ha)}{dh} + \frac{\partial f}{\partial y}(x_0 + ha, y_0 + hb)\frac{d(y_0 + hb)}{dh} \\ &= a\frac{\partial f}{\partial x}(x_0 + ha, y_0 + hb) + b\frac{\partial f}{\partial y}(x_0 + ha, y_0 + hb) \end{aligned}$$

when $h = 0$, we get

$$g'(0) = a\frac{\partial f}{\partial x}(x_0, y_0) + b\frac{\partial f}{\partial y}(x_0, y_0)$$

Combining the above results, we conclude

$$D_{\vec{u}} f(x_0, y_0) = a\frac{\partial f}{\partial x}(x_0, y_0) + b\frac{\partial f}{\partial y}(x_0, y_0)$$

□

Remark 2. If the unit vector $\vec{u}$ forms an angle θ with the positive x-axis, then $\vec{u} = (\cos\theta, \sin\theta)$. We can compute the directional derivative by

$$D_{\vec{u}}f(x_0, y_0) = \frac{\partial f}{\partial x}(x_0, y_0)\cos\theta + \frac{\partial f}{\partial y}(x_0, y_0)\sin\theta$$

Example 3. *Find the directional derivative* $D_{\vec{u}}$ *if* $f(x, y) = x^3 - 3xy + 4y^2$*, and* $\vec{u}$ *is the unit vector given by angle* $\theta = \frac{\pi}{6}$*. What is* $D_{\vec{u}}f(1, 2)$*?*

$$D_{\vec{u}}f(x, y) = \frac{\partial f}{\partial x}(x, y)\cos\frac{\pi}{6} + \frac{\partial f}{\partial y}(x, y)\sin\frac{\pi}{6} = (3x^2 - 3y)\frac{\sqrt{3}}{2} + (-3x + 8y)\frac{1}{2}$$

and

$$D_{\vec{u}}f(1, 2) = \frac{13 - 3\sqrt{3}}{2}$$

2 Gradient

In the previous section, we have seen that if a unit vector $\vec{u} = (a, b)$, the directional derivative of f along $\vec{u}$ is given by

$$D_{\vec{u}}f(x, y) = a\frac{\partial f}{\partial x}(x, y) + b\frac{\partial f}{\partial y}(x, y)$$

We can rewrite it in the following form as a dot product:

$$D_{\vec{u}}f(x, y) = (\frac{\partial f}{\partial x}(x, y), \frac{\partial f}{\partial y}(x, y)).(a, b)$$

Definition 4. The gradient of a function $f(x, y)$ is $\nabla f(x, y) = (\frac{\partial f}{\partial x}(x, y), \frac{\partial f}{\partial y}(x, y))$

By this definition, we can write

$$D_{\vec{u}}f(x, y) = \nabla f(x, y).\vec{u}$$

Example 5. *Find the gradient of the function* $f(x, y) = x^2y^3 - 4y$ *at* $(2, -1)$*, and find the directional derivative in the direction of the vector* $\vec{v} = (2, 5)$*.*

$\nabla f(x, y) = (\frac{\partial f}{\partial x}(x, y), \frac{\partial f}{\partial y}(x, y)) = (2xy^3, 3x^2y^2 - 4)$, *so* $\nabla f(2, -1) = (-4, 8)$.

Note that $\vec{v}$ is not a unit vector, so we first compute the unit vector in the direction of $\vec{v}$, which is $\frac{\vec{v}}{|\vec{v}|} = (\frac{2}{\sqrt{29}}, \frac{5}{\sqrt{29}})$. So

$$D_{\frac{\vec{v}}{|\vec{v}|}} f(2,-1) = \nabla f(2,-1).\frac{\vec{v}}{|\vec{v}|} = (-4,8).(\frac{2}{\sqrt{29}}, \frac{5}{\sqrt{29}}) = \frac{32}{\sqrt{29}}$$

Theorem 6. *If $f(x,y)$ is a differentiable function, then the maximum value of $D_{\vec{u}}f(x_0,y_0)$ is $|\nabla f(x_0,y_0)|$, and it is achieved when $\vec{u} = \frac{\nabla f(x_0,y_0)}{|\nabla f(x_0,y_0)|}$*

Proof. $D_{\vec{u}}f(x_0,y_0) = \nabla f(x_0,y_0).\vec{u} = |\nabla f(x_0,y_0)||\vec{u}|\cos\theta = |\nabla f(x_0,y_0)|\cos\theta$, so it achieves maximum when $\theta = 0$. $\square$

Example 7. $f(x,y) = xe^y$. *In which direction does f have the maximum rate of change at $(2,0)$? What is the maximum rate of change?*

$\nabla f(x,y) = (e^y, xe^y)$, *so* $\nabla f(2,0) = (1,2)$. *The maximum rate of change is along the direction of $(1,2)$ and the maximum rate of change is $|(1,2)| = \sqrt{5}$*

3 Level Set

Definition 8. $f(x,y)$ is a function, and c is a real number. Define the set $\{(x,y) \in \mathbb{R}^2 | f(x,y) = c\}$ to be the level set of f corresponding to the value c.

In other words, the level set of $f(x,y)$ corresponding to the value c is the set of points (x,y) at which the value of f is c.

Example 9. $f(x,y) = x+y$. *The level set $f(x,y) = 0$ is the set of points satisfying $x+y=0$, i.e. the straight line $y=-x$.*

Example 10. *Let $f(x,y) = x^2+y^2$, then the level set $f(x,y)=1$ is the unit circle centred at origin.*

Exercise 11. *Sketch the level set of the function $f(x,y) = xy = c$. (You may need to discuss the cases $c>0, c=0, c<0$ separately.)*

Example 12. *In geology, the altitude is a function of the location on earth. People often use a topographic map to describe altitude by sketching some level sets, in which case is often a curve. When the curves are denser, it means the area is steeper. If we travel along a level curve, the altitude doesn't change.*

It turns out there is a close relation between level sets and gradient of a function:

Theorem 13. *$f(x, y)$ is a differentiable function. If $f(x, y) = c$ is a level set and $\nabla f(x, y) \neq (0, 0)$, then $\nabla f(x, y)$ is perpendicular to the tangent line of $f(x, y) = c$ at (x, y).*

Coming back to the previous example regarding topographic maps, the above theorem indicates that if we want to climb onto a mountain in a shortest path, we should always go in the direction perpendicular to the level curve, since this is the direction of the gradient.

Remark 14. The concept of level sets also applies to functions of more variables. For example, $f(x, y, z) = x^2 + y^2 + z^2$, the level set $f(x, y, z) = 1$ is the unit sphere centred at origin.