Directional Derivative, Gradient and Level Set

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1 Directional Derivative

The partial derivatives of a multi-variable function \( f(x, y) \), \( \frac{\partial f}{\partial x} \) and \( \frac{\partial f}{\partial y} \), tell us the rate of change of the function along the \( x \)-axis and \( y \)-axis respectively. But in general what about the rate of change in other directions?

On the \( xy \)-plane, each direction can be represented by a unit vector \( \vec{u} \).

We are going to define the directional derivative of a function \( z = f(x, y) \) at \((x_0, y_0)\) in the direction \( \vec{u} \):

On the \( xy \)-plane, consider the line \( l \) passing through \((x_0, y_0)\) and parallel to the unit vector \( \vec{u} \). Passing through the line \( l \), there is a unique vertical plane \( \alpha \), and \( \alpha \) intersects the graph of \( z = f(x, y) \) along a curve \( C \), so \( C \) projects to \( l \) on the \( xy \)-plane.

If we start at \((x_0, y_0)\) and travel along \( \vec{u} \) direction for a distance \( h \), arriving at \((x, y)\). Then the vector with initial point \((x_0, y_0)\) and terminal point \((x, y)\) is \((x - x_0, y - y_0) = h\vec{u}\). Since \( \vec{u} \) is a unit vector, let \( \vec{u} = (a, b) \), where \( a^2 + b^2 = 1 \). So:

\[
(x - x_0, y - y_0) = h\vec{u} = (ha, hb)
\]

which implies

\[
x = x_0 + ha, \ y = y_0 + hb
\]

so the rate of change of \( f \) along \( \vec{u} \) at \((x_0, y_0)\) is

\[
D_{\vec{u}}f(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}
\]

We call it the directional derivative of \( f \) at \((x_0, y_0)\) in the direction of \( \vec{u} \).

Indeed there is a faster way to evaluate the directional derivative.
Theorem 1. If $f$ is a differentiable function of $x$ and $y$, then $f$ has a directional derivative in the direction of any unit vector $\vec{u} = (a, b)$ and

$$D_{\vec{u}} f(x_0, y_0) = a \frac{\partial f}{\partial x}(x_0, y_0) + b \frac{\partial f}{\partial y}(x_0, y_0)$$

Proof. Define $g(h) = f(x_0 + ha, y_0 + hb)$. We get

$$g'(0) = \lim_{h \to 0} \frac{g(h) - g(0)}{h} = \lim_{h \to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h} = D_{\vec{u}} f(x_0, y_0)$$

On the other hand, by the chain rule,

$$g'(h) = \frac{\partial f}{\partial x}(x_0 + ha, y_0 + hb) \frac{d(x_0 + ha)}{dh} + \frac{\partial f}{\partial y}(x_0 + ha, y_0 + hb) \frac{d(y_0 + hb)}{dh}$$

$$= a \frac{\partial f}{\partial x}(x_0 + ha, y_0 + hb) + b \frac{\partial f}{\partial y}(x_0 + ha, y_0 + hb)$$

when $h = 0$, we get

$$g'(0) = a \frac{\partial f}{\partial x}(x_0, y_0) + b \frac{\partial f}{\partial y}(x_0, y_0)$$

Combining the above results, we conclude

$$D_{\vec{u}} f(x_0, y_0) = a \frac{\partial f}{\partial x}(x_0, y_0) + b \frac{\partial f}{\partial y}(x_0, y_0)$$
Remark 2. If the unit vector \( \vec{u} \) forms an angle \( \theta \) with the positive \( x \)-axis, then \( \vec{u} = (\cos \theta, \sin \theta) \). We can compute the directional derivative by

\[
D_{\vec{u}}f(x_0, y_0) = \frac{\partial f}{\partial x}(x_0, y_0) \cos \theta + \frac{\partial f}{\partial y}(x_0, y_0) \sin \theta
\]

Example 3. Find the directional derivative \( D_{\vec{u}} \) if \( f(x, y) = x^3 - 3xy + 4y^2 \), and \( \vec{u} \) is the unit vector given by angle \( \theta = \frac{\pi}{6} \). What is \( D_{\vec{u}}f(1, 2) \)?

\[
D_{\vec{u}}f(x, y) = \frac{\partial f}{\partial x}(x, y) \cos \frac{\pi}{6} + \frac{\partial f}{\partial y}(x, y) \sin \frac{\pi}{6} = (3x^2 - 3y)\frac{\sqrt{3}}{2} + (-3x + 8y)\frac{1}{2}
\]

and

\[
D_{\vec{u}}f(1, 2) = \frac{13 - 3\sqrt{3}}{2}
\]

2 Gradient

In the previous section, we have seen that if a unit vector \( \vec{u} = (a, b) \), the directional derivative of \( f \) along \( \vec{u} \) is given by

\[
D_{\vec{u}}f(x, y) = a \frac{\partial f}{\partial x}(x, y) + b \frac{\partial f}{\partial y}(x, y)
\]

We can rewrite it in the following form as a dot product:

\[
D_{\vec{u}}f(x, y) = (\frac{\partial f}{\partial x}(x, y), \frac{\partial f}{\partial y}(x, y)).(a, b)
\]

Definition 4. The gradient of a function \( f(x, y) \) is \( \nabla f(x, y) = (\frac{\partial f}{\partial x}(x, y), \frac{\partial f}{\partial y}(x, y)) \)

By this definition, we can write

\[
D_{\vec{u}}f(x, y) = \nabla f(x, y).\vec{u}
\]

Example 5. Find the gradient of the function \( f(x, y) = x^2y^3 - 4y \) at \((2, -1)\), and find the directional derivative in the direction of the vector \( \vec{v} = (2, 5) \).

\[
\nabla f(x, y) = (\frac{\partial f}{\partial x}(x, y), \frac{\partial f}{\partial y}(x, y)) = (2xy^3, 3x^2y^2 - 4), \text{ so } \nabla f(2, -1) = (-4, 8).
\]
Note that \( \vec{v} \) is not a unit vector, so we first compute the unit vector in the direction of \( \vec{v} \), which is \( \frac{\vec{v}}{|\vec{v}|} = \left( \frac{2}{\sqrt{29}}, \frac{5}{\sqrt{29}} \right) \). So

\[
D_{\frac{\vec{v}}{|\vec{v}|}} f(2, -1) = \nabla f(2, -1) \cdot \frac{\vec{v}}{|\vec{v}|} = (-4, 8) \cdot \left( \frac{2}{\sqrt{29}}, \frac{5}{\sqrt{29}} \right) = \frac{32}{\sqrt{29}}
\]

**Theorem 6.** If \( f(x, y) \) is a differentiable function, then the maximum value of \( D_{\vec{u}} f(x_0, y_0) \) is \( |\nabla f(x_0, y_0)| \), and it is achieved when \( \vec{u} = \frac{\nabla f(x_0, y_0)}{|\nabla f(x_0, y_0)|} \).

Proof. \( D_{\vec{u}} f(x_0, y_0) = \nabla f(x_0, y_0) \cdot \vec{u} = |\nabla f(x_0, y_0)||\vec{u}| \cos \theta = |\nabla f(x_0, y_0)| \cos \theta \), so it achieves maximum when \( \theta = 0 \).

**Example 7.** \( f(x, y) = xe^y \). In which direction does \( f \) have the maximum rate of change at \((2, 0)\)? What is the maximum rate of change?

\[
\nabla f(x, y) = (e^y, xe^y), \text{ so } \nabla f(2, 0) = (1, 2). \text{ The maximum rate of change is along the direction of } (1, 2) \text{ and the maximum rate of change is } |(1, 2)| = \sqrt{5}
\]

### 3 Level Set

**Definition 8.** \( f(x, y) \) is a function, and \( c \) is a real number. Define the set \( \{(x, y) \in \mathbb{R}^2 | f(x, y) = c \} \) to be the level set of \( f \) corresponding to the value \( c \).

In other words, the level set of \( f(x, y) \) corresponding to the value \( c \) is the set of points \((x, y)\) at which the value of \( f \) is \( c \).

**Example 9.** \( f(x, y) = x + y \). The level set \( f(x, y) = 0 \) is the set of points satisfying \( x + y = 0 \), i.e. the straight line \( y = -x \).

**Example 10.** Let \( f(x, y) = x^2 + y^2 \), then the level set \( f(x, y) = 1 \) is the unit circle centred at origin.

**Exercise 11.** Sketch the level set of the function \( f(x, y) = xy = c \). (You may need to discuss the cases \( c > 0, c = 0, c < 0 \) separately.)

**Example 12.** In geology, the altitude is a function of the location on earth. People often use a topographic map to describe altitude by sketching some level sets, in which case is often a curve. When the curves are denser, it means the area is steeper. If we travel along a level curve, the altitude doesn’t change.
It turns out there is a close relation between level sets and gradient of a function:

**Theorem 13.** \( f(x, y) \) is a differentiable function. If \( f(x, y) = c \) is a level set and \( \nabla f(x, y) \neq (0, 0) \), then \( \nabla f(x, y) \) is perpendicular to the tangent line of \( f(x, y) = c \) at \((x, y)\).

Coming back to the previous example regarding topographic maps, the above theorem indicates that if we want to climb onto a mountain in a shortest path, we should always go in the direction perpendicular to the level curve, since this is the direction of the gradient.

**Remark 14.** The concept of level sets also applies to functions of more variables. For example, \( f(x, y, z) = x^2 + y^2 + z^2 \), the level set \( f(x, y, z) = 1 \) is the unit sphere centred at origin.