Homework V Solution

First-Half

1. $\frac{\partial z}{\partial x} = -\frac{1}{2}(x^2+y^2)^{-\frac{3}{2}}2x = -\frac{x}{(x^2+y^2)^{\frac{3}{2}}}$,

   $\frac{\partial z}{\partial y} = -\frac{1}{2}(x^2+y^2)^{-\frac{3}{2}}2y = -\frac{y}{(x^2+y^2)^{\frac{3}{2}}}$

   So $El_x z = \frac{x}{z}\frac{\partial z}{\partial x} = \frac{x}{(x^2+y^2)^{-\frac{1}{2}}}(-\frac{x}{(x^2+y^2)^{\frac{3}{2}}}) = -\frac{x^2}{x^2+y^2}$

   $El_x z = \frac{y}{z}\frac{\partial z}{\partial y} = \frac{y}{(x^2+y^2)^{-\frac{1}{2}}}(-\frac{y}{(x^2+y^2)^{\frac{3}{2}}}) = -\frac{y^2}{x^2+y^2}$

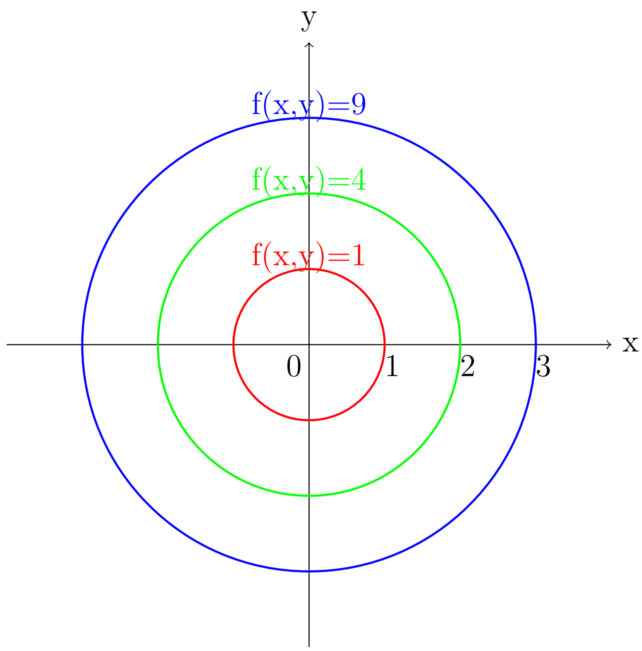
2. Since x and y are the populations of two cities, they are positive. d is the distance between the two cities, so it is also positive.

   $\frac{\partial T}{\partial x} = \frac{ky}{d^n} > 0$
   $\frac{\partial T}{\partial y} = \frac{kx}{d^n} > 0$
   $\frac{\partial T}{\partial d} = -\frac{nkxy}{d^{n+1}} < 0$

3. $\frac{\partial w}{\partial x} = 3yz + 2xy - z^3$, $\frac{\partial w}{\partial y} = 3xz + x^2$, $\frac{\partial w}{\partial z} = 3xy - 3xz^2$

$\frac{\partial^2 w}{\partial x^2} = 2y$	$\frac{\partial^2 w}{\partial x \partial y} = 3z + 2x$	$\frac{\partial^2 w}{\partial x \partial z} = 3y - 3z^2$
$\frac{\partial^2 w}{\partial y \partial x} = 3z + 2x$	$\frac{\partial^2 w}{\partial y^2} = 0$	$\frac{\partial^2 w}{\partial y \partial z} = 3x$
$\frac{\partial^2 w}{\partial z \partial x} = 3y - 3z^2$	$\frac{\partial^2 w}{\partial z \partial y} = 3x$	$\frac{\partial^2 w}{\partial z^2} = -6xz$

4.



5. An example is $F(x, y) = xy$.

6. Let $F(x, y) = 3x^2 - 3xy^2 + y^3 + 3y^2$, then

$$\frac{dy}{dx} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}} = -\frac{6x - 3y^2}{-6xy + 3y^2 + 6y}$$

So $\frac{dy}{dx}(1, 1) = -\frac{6-3}{-6+3+6} = -1$, and the tangent line is

$$y - 1 = -(x - 1)$$

7. Let $F(x, y, z) = x^3 + y^3 + z^3 - 3z$, then

$$\frac{\partial z}{\partial x} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}} = -\frac{3x^2}{3z^2 - 3}$$

$$\frac{\partial z}{\partial y} = -\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}} = -\frac{3y^2}{3z^2 - 3}$$

8. $R_{yx} = \frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}} = \frac{ax^{a-1}}{ay^{a-1}} = \frac{x^{a-1}}{y^{a-1}}$

9. $R_{yx} = \frac{x^{a-1}}{y^{a-1}} = (\frac{x}{y})^{a-1}$, so $(\frac{y}{x})^{a-1} = R_{yx}^{-1}$, $\frac{y}{x} = R_{yx}^{-\frac{1}{a-1}}$

$$\sigma_{yx} = El_{R_{yx}}(\frac{y}{x}) = \frac{R_{yx}}{R_{yx}^{-\frac{1}{a-1}}} \frac{d(\frac{y}{x})}{dR_{yx}} = \frac{R_{yx}}{R_{yx}^{-\frac{1}{a-1}}}(-\frac{1}{a-1} R_{yx}^{-\frac{1}{a-1}-1}) = -\frac{1}{a-1}$$

Second-Half

1. $f(0,0) = 0$, $\frac{\partial f}{\partial x} = e^x \ln(1+y)$, $\frac{\partial f}{\partial y} = \frac{e^x}{1+y}$, so $\frac{\partial f}{\partial x}(0,0) = 0$ and $\frac{\partial f}{\partial y}(0,0) = 1$

   So near $(0,0)$,

$$f(x,y) \approx f(0,0) + \frac{\partial f}{\partial x}(0,0)(x-0) + \frac{\partial f}{\partial y}(0,0)(y-0) = y$$

2. $f(1,2) = 1$, $\frac{\partial f}{\partial x} = 6x + y$, $\frac{\partial f}{\partial y} = x - 2y$, so $\frac{\partial f}{\partial x}(1,2) = 8$ and $\frac{\partial f}{\partial y}(1,2) = -3$

   So the linear approximation near $(1,2)$ is

$$f(x,y) \approx 1 + 8(x-1) - 3(y-2)$$

   So $f(1.02, 1.99) \approx 1 + 8(1.02-1) - 3(1.99-2) = 1 + 0.16 + 0.03 = 1.19$

3. Let $F(x,y,z) = x^2 + y^2 + z^2$.

   $\frac{\partial z}{\partial x} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}} = -\frac{2x}{2z} = -\frac{x}{z}$

   $\frac{\partial z}{\partial y} = -\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}} = -\frac{2y}{2z} = -\frac{y}{z}$

   When $(x,y,z) = (1,1,1)$, $\frac{\partial z}{\partial x} = -1$, $\frac{\partial z}{\partial y} = -1$. So the tangent plane is

$$z - 1 = -(x-1) - (y-1)$$

   i.e. $x + y + z = 3$

4. Method I:

$$\begin{aligned}
Ue^U &= x\sqrt{y} \\
dUe^U &= dx\sqrt{y} \\
Ude^U + e^U dU &= xd\sqrt{y} + \sqrt{y}dx \\
Ue^U dU + e^U dU &= x\frac{1}{2\sqrt{y}}dy + \sqrt{y}dx \\
dU &= \frac{1}{(U+1)e^U}(\frac{x}{2\sqrt{y}}dy + \sqrt{y}dx)
\end{aligned}$$

Method II:

$$\begin{aligned}
Ue^U &= x\sqrt{y} \\
\ln Ue^U &= \ln xy^{\frac{1}{2}} \\
\ln U + U &= \ln x + \frac{1}{2}\ln y \\
d(\ln U + U) &= d(\ln x + \frac{1}{2}\ln y) \\
d\ln U + dU &= d\ln x + \frac{1}{2}d\ln y \\
\frac{1}{U}dU + dU &= \frac{1}{x}dx + \frac{1}{2y}dy \\
\frac{1+U}{U}dU &= \frac{1}{x}dx + \frac{1}{2y}dy \\
dU &= \frac{U}{1+U}(\frac{1}{x}dx + \frac{1}{2y}dy)
\end{aligned}$$

Remark 1. Though at first glance the two methods lead to different answers, but indeed they are the same: Since $Ue^U = x\sqrt{y}$, $\frac{1}{e^U} = \frac{U}{x\sqrt{y}}$

$\frac{1}{(U+1)e^U}(\frac{x}{2\sqrt{y}}dy + \sqrt{y}dx) = \frac{U}{(U+1)x\sqrt{y}}(\frac{x}{2\sqrt{y}}dy + \sqrt{y}dx) = \frac{U}{U+1}(\frac{1}{2y}dy + \frac{1}{x}dx)$

5. $x = 108 - 3y - 4z$, so

$$U = xyz = (108 - 3y - 4z)yz = 108yz - 3y^2z - 4yz^2$$

$\frac{\partial U}{\partial y} = 108z - 6yz - 4z^2 = z(108 - 6y - 4z)$
$\frac{\partial U}{\partial z} = 108y - 3y^2 - 8yz = y(108 - 3y - 8z)$.
Let $\frac{\partial U}{\partial y} = 0$ and $\frac{\partial U}{\partial z} = 0$, together with the assumption that x, y, z are positive, we get $y = 12, z = 9$, so $x = 108 - 3 \times 12 - 4 \times 9 = 36$

Since the question assumes we can get maximum at the critical point, the utility is maximized when $x = 36, y = 12, z = 9$

6. The profit function is

$$\pi(x, y) = px + qy - C(x, y) = px + qy - x^2 - xy - y^2 - x - y - 14$$

$\frac{\partial \pi}{\partial x} = p - 2x - y - 1 = -2x - y + (p - 1)$
$\frac{\partial \pi}{\partial y} = q - x - 2y - 1 = -x - 2y + (q - 1)$
Let $\frac{\partial \pi}{\partial x} = \frac{\partial \pi}{\partial y} = 0$, we get $x = \frac{2p-q-1}{3}$ and $y = \frac{2q-p-1}{3}$.

$\frac{\partial^2 \pi}{\partial x^2} = -2 < 0$, $\frac{\partial^2 \pi}{\partial y^2} = -2 < 0$.

$\frac{\partial^2 \pi}{\partial x \partial y} = -1$, so $\frac{\partial^2 \pi}{\partial x^2}\frac{\partial^2 \pi}{\partial y^2} - (\frac{\partial^2 \pi}{\partial x \partial y})^2 = 3 > 0$
So $(\frac{2p-q-1}{3}, \frac{2q-p-1}{3})$ is a maximal point.

7. $\frac{\partial f}{\partial x} = -4x + 4$
$\frac{\partial f}{\partial y} = -2y + 4$
Let $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0$, we get $x = 1, y = 2$, so the extreme point is $(1, 2)$

$\frac{\partial^2 f}{\partial x^2} = -4 < 0$, $\frac{\partial^2 f}{\partial x^2} = -2 < 0$

$\frac{\partial^2 f}{\partial x \partial y} = 0$, so $\frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2} - (\frac{\partial^2 f}{\partial x \partial y})^2 = 8 > 0$
So $(1, 2)$ is a maximum point

8. $\frac{\partial f}{\partial x} = 2x + 2y^2$
$\frac{\partial f}{\partial y} = 4xy + 4y$
Let $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0$, we get the critical points are $(0, 0), (-1, 1), (-1, -1)$

$\frac{\partial^2 f}{\partial x^2} = 2$, $\frac{\partial^2 f}{\partial y^2} = 4x + 4$

$\frac{\partial^2 f}{\partial x \partial y} = 4y$, so $H = \frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2} - (\frac{\partial^2 f}{\partial x \partial y})^2 = 2(4x+4) - (4y)^2 = 8(x+1-2y^2)$

At $(0, 0)$, $\frac{\partial^2 f}{\partial x^2} = 2 > 0$ and $H = 8 > 0$, so $(0, 0)$ is a local minimum.
At $(-1, 1)$, $H = -16 < 0$, so $(-1, 1)$ is a saddle point
At $(-1, -1)$, $H = -16 < 0$, so $(-1, -1)$ is a saddle point