Homework IV Solution

First-Half

1. $h'(t) = \frac{1}{2}\frac{1}{\sqrt{t}} - \frac{1}{2} = \frac{1}{2}(\frac{1}{\sqrt{t}} - 1)$. Let $h'(t) = 0$, we get $t = 1$.

   $h''(t) = \frac{1}{2}(-\frac{1}{2}t^{-\frac{3}{2}}) = -\frac{1}{4}t^{-\frac{3}{2}} \leq 0$ on $[0, 3]$, so the function is concave on $[0.3]$.

   We conclude that $t = 1$ is a maximum point, so at time $t = 1$ the plant is tallest.

2. $f'(x) = 2x + 3$. $f'(x) = 0$ implies $x = -\frac{3}{2}$. So the extreme point is $x = -\frac{3}{2}$.

   $f''(x) = 2 > 0$, so f is convex, the extreme point is a minimum point.

3. $f'(x) = 4x^3 - 12x^2 + 8x = 4x(x^2 - 3x + 2) = 4x(x - 1)(x - 2)$. So $f'(x) = 0$ when $x = 0, 1, 2$.

   The candidates are $x = -1, 0, 1, 2, 3$.

   $f(-1) = 9,\ f(0) = 0,\ f(1) = 1,\ f(2) = 0,\ f(3) = 9$.

   So the maximum points are $x = -1, 3$, the maximum value is 9. The minimum points are $x = 0, 2$, the minimum value is 0.

4.

$$\begin{aligned} f'(x) &= \frac{4x(x^4+1) - 2x^2(4x^3)}{(x^4+1)^2} \\ &= \frac{4x - 4x^5}{(x^4+1)^2} \\ &= \frac{4x(1-x^4)}{(x^4+1)^2} \\ &= \frac{4x(1+x^2)(1-x^2)}{(x^4+1)^2} \\ &= \frac{4x(1+x^2)(1+x)(1-x)}{(x^4+1)^2} \end{aligned}$$

Let $f'(x) = 0$, we get $x = -1, 0, 1$.

So the candidates are $x = -2, -1, 0, 1, 2$.

$f(-2) = \frac{8}{17}$, $f(-1) = 1$, $f(0) = 0$, $f(1) = 1$, $f(2) = \frac{8}{17}$

So the maximum value is 1.

5. $f'(x) = 1 - \frac{1}{x^2}$. Let $f'(x) = 0$, we get $x = 1$ or $x = -1$.

   $f''(x) = \frac{2}{x^3}$, so $f''(-1) = -2 < 0$, -1 is a local maximum point; $f''(1) = 2 > 0$, 1 is a local minimum point.

6. $f'(x) = 2xe^{x^2} - 2xe^{2-x^2}$. Let $f'(x) = 0$, we get

$$2xe^{x^2} - 2xe^{2-x^2} = 2x(e^{x^2} - e^{2-x^2}) = 0$$

So $x = 0$ or $e^{x^2} - e^{2-x^2} = 0$. The latter holds if and only if $x^2 = 2 - x^2$, i.e. $x^2 = 1$, so $x = \pm 1$.

So the critical points are $x = -1, 0, 1$.

The sign of $f'(x)$ is as follows:

x	$(-\infty, -1)$	$(-1, 0)$	$(0, 1)$	$(1, +\infty)$
$f'(x)$	< 0	> 0	< 0	> 0

So $x = -1$ and $x = 1$ are local minimum points, and $x = 0$ is local maximum point.

7. Suppose one of the edges of the rectangle has length x, then the area of the rectangle is given by $A = A(x) = x(\frac{l}{2} - x) = \frac{lx}{2} - x^2$, with $x \in (0, \frac{l}{2})$.

   $A'(x) = \frac{l}{2} - 2x$, let $A'(x) = 0$, we get $x = \frac{l}{4}$.

   $A''(x) = -2$, the function is concave, so $x = \frac{l}{4}$ is a maximal point. The largest possible area, i.e. the maximal value is $A = A(\frac{l}{4}) = \frac{l^2}{16}$

8. $\bar{T} = \frac{T}{W} = \frac{a(bW+c)^p}{W} + k$

   So

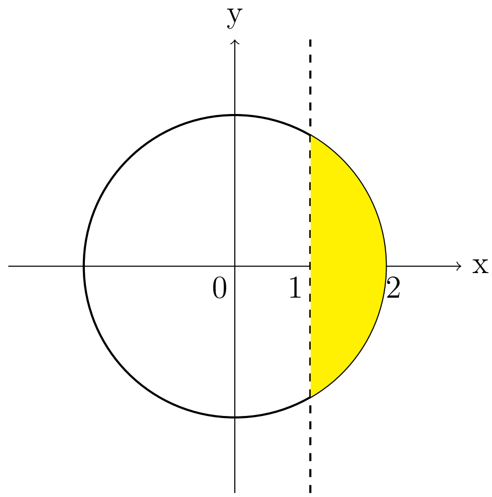
$$\begin{aligned}
\bar{T}' &= \frac{abp(bW+c)^{p-1}W - a(bW+c)^p}{W^2} \\
&= \frac{a(bW+c)^{p-1}[bpW - (bW+c)]}{W^2} \\
&= \frac{a(bW+c)^{p-1}[b(p-1)W - c]}{W^2} \\
&= \frac{a(bW+c)^{p-1}b(p-1)(W - \frac{c}{b(p-1)})}{W^2}
\end{aligned}$$

   Since $W > 0$, the only case when W is a critical point is $W = \frac{c}{b(p-1)}$.

   Note that when $W < \frac{c}{b(p-1)}$, $\bar{T}' < 0$, and when $W > \frac{c}{b(p-1)}$, $\bar{T}' > 0$, so $W = \frac{c}{b(p-1)}$ is a minimum point. When $x = \frac{c}{b(p-1)}$, the average tax is minimized.

Second-Half

1. $x - 1 > 0$ and $4 - x^2 - y^2 \geq 0$, so we get $x > 1$ and $x^2 + y^2 \leq 4$. The domain is $\{(x, y) \in \mathbb{R}^2 | x > 1 \text{ and } x^2 + y^2 \leq 4\}$.



2. $e^{x+y} - 1 \neq 0$, so $e^{x+y} \neq 1$, i.e. $x + y \neq 0$.

   So the domain is $\{(x, y) \in \mathbb{R}^2 | x + y \neq 0\}$

3. (a). Since this function only depends on x, $\frac{\partial f}{\partial x} = \frac{df}{dx} = 8x^7 e^{3x} + x^8(3e^{3x}) = (8x^7 + 3x^8)e^{3x}$, and $\frac{\partial f}{\partial y} = 0$

   (b). $\frac{\partial f}{\partial x} = \frac{1}{\sqrt{y^2+1}} \frac{1}{x} = \frac{1}{x\sqrt{y^2+1}}$

   $\frac{\partial f}{\partial y} = (\ln x)(-\frac{1}{2}(y^2 + 1)^{-\frac{3}{2}} \times 2y) = \frac{-y \ln x}{(y^2+1)^{\frac{3}{2}}}$

4. (a). $\frac{\partial f}{\partial x} = \frac{(x+y)-(x-y)}{(x+y)^2} = \frac{2y}{(x+y)^2}$

   $\frac{\partial f}{\partial y} = \frac{-(x+y)-(x-y)}{(x+y)^2} = \frac{-2x}{(x+y)^2}$

   So $\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}(\frac{\partial f}{\partial x}) = \frac{\partial}{\partial x}(\frac{2y}{(x+y)^2}) = (2y)(-2(x + y)^{-3}) = \frac{-4y}{(x+y)^3}$

   $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}(\frac{\partial f}{\partial y}) = \frac{\partial}{\partial x}(\frac{-2x}{(x+y)^2}) = -2\frac{(x+y)^2 - x(2(x+y))}{(x+y)^4} = \frac{-2(y^2-x^2)}{(x+y)^4} = \frac{2(x-y)}{(x+y)^3}$

   $\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}(\frac{\partial f}{\partial x}) = \frac{\partial}{\partial y}(\frac{2y}{(x+y)^2}) = 2\frac{(x+y)^2 - y(2(x+y))}{(x+y)^4} = \frac{2(x^2-y^2)}{(x+y)^4} = \frac{2(x-y)}{(x+y)^3}$

   $\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}(\frac{\partial f}{\partial y}) = \frac{\partial}{\partial y}(\frac{-2x}{(x+y)^2}) = (-2x)(-2(x + y)^{-3}) = \frac{4x}{(x+y)^3}$

(b). $\frac{\partial f}{\partial x} = \frac{1}{2}(x^2+y^2)^{-\frac{1}{2}}(2x) = \frac{x}{\sqrt{x^2+y^2}}$

$\frac{\partial f}{\partial y} = \frac{1}{2}(x^2+y^2)^{-\frac{1}{2}}(2y) = \frac{y}{\sqrt{x^2+y^2}}$

$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}(\frac{\partial f}{\partial x}) = \frac{\partial}{\partial x}(\frac{x}{\sqrt{x^2+y^2}}) = \frac{\sqrt{x^2+y^2}-x\frac{x}{\sqrt{x^2+y^2}}}{x^2+y^2} = \frac{\frac{x^2+y^2-x^2}{\sqrt{x^2+y^2}}}{x^2+y^2} = \frac{y^2}{(x^2+y^2)^{\frac{3}{2}}}$

$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}(\frac{\partial f}{\partial y}) = \frac{\partial}{\partial x}(\frac{y}{\sqrt{x^2+y^2}}) = y(-\frac{1}{2}(x^2+y^2)^{-\frac{3}{2}})(2x) = \frac{-xy}{(x^2+y^2)^{\frac{3}{2}}}$

$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}(\frac{\partial f}{\partial x}) = \frac{\partial}{\partial y}(\frac{x}{\sqrt{x^2+y^2}}) = x(-\frac{1}{2}(x^2+y^2)^{-\frac{3}{2}})(2y) = \frac{-xy}{(x^2+y^2)^{\frac{3}{2}}}$

$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}(\frac{\partial f}{\partial y}) = \frac{\partial}{\partial y}(\frac{y}{\sqrt{x^2+y^2}}) = \frac{\sqrt{x^2+y^2}-y\frac{y}{\sqrt{x^2+y^2}}}{x^2+y^2} = \frac{\frac{x^2+y^2-y^2}{\sqrt{x^2+y^2}}}{x^2+y^2} = \frac{x^2}{(x^2+y^2)^{\frac{3}{2}}}$

(c). $\frac{\partial f}{\partial x} = \frac{y}{x}$, $\frac{\partial f}{\partial y} = \ln x$

$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}(\frac{\partial f}{\partial x}) = \frac{\partial}{\partial x}(\frac{y}{x}) = -\frac{y}{x^2}$

$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}(\frac{\partial f}{\partial y}) = \frac{\partial}{\partial x}(\ln x) = \frac{1}{x}$

$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}(\frac{\partial f}{\partial x}) = \frac{\partial}{\partial y}(\frac{y}{x}) = \frac{1}{x}$

$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}(\frac{\partial f}{\partial y}) = \frac{\partial}{\partial y}(\ln x) = 0$

5.

$$\begin{aligned}
\frac{dz}{dt} &= \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} \\
&= (px^{p-1}y^q)(a) + (qx^p y^{q-1})(b) \\
&= ap(at)^{p-1}(bt)^q + bq(at)^p(bt)^{q-1} \\
&= pa^p b^q t^{p+q-1} + qa^p b^q t^{p+q-1} \\
&= (p+q)a^p b^q t^{p+q-1}
\end{aligned}$$

6.

$$\begin{aligned}
\frac{\partial z}{\partial t} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} \\
&= (4x)(2t) + 6y \\
&= 8(t^2 - s)t + 6(t + 2s^3) \\
&= 8t^3 - 8st + 6t + 12s^3
\end{aligned}$$

$$
\begin{aligned}
\frac{\partial z}{\partial s} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} \\
&= (4x)(-1) + (6y)(6s^2) \\
&= -4(t^2 - s) + 36(t + 2s^3)s^2 \\
&= -4t^2 + 4s + 36ts^2 + 72s^5
\end{aligned}
$$

7.

$$
\begin{aligned}
\frac{\partial z}{\partial t} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} \\
&= (e^{x-y} + \frac{1}{x+y}) + (-e^{x-y} + \frac{1}{x+y}) \\
&= \frac{2}{x+y} \\
&= \frac{1}{t}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial z}{\partial s} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} \\
&= (e^{x-y} + \frac{1}{x+y}) - (-e^{x-y} + \frac{1}{x+y}) \\
&= 2e^{x-y} \\
&= 2e^{2s}
\end{aligned}
$$

So $\frac{\partial z}{\partial t}(1,0) = \frac{1}{1} = 1$, $\frac{\partial z}{\partial s}(1,0) = 2e^0 = 2$

8. $\frac{dD}{dt} = \frac{\partial D}{\partial x}\frac{dx}{dt} + \frac{\partial D}{\partial y}\frac{dy}{ds} = \frac{x}{\sqrt{x^2+y^2}}(2t) + \frac{y}{\sqrt{x^2+y^2}}(3t^2+1)$

When $t = 2$, $x = 2^2 = 4$ and $y = 2^3 + 2 = 10$, so
$V_r(2) = \frac{dD}{dt}(2) = \frac{4}{\sqrt{4^2+10^2}}(2 \times 2) + \frac{10}{\sqrt{4^2+10^2}}(3 \times 2^2 + 1) = \frac{146}{\sqrt{116}} = \frac{73}{\sqrt{29}} = \frac{73}{29}\sqrt{29}$