Homework II Solution

First-Half

1. $f'(x) = -\frac{1}{2}(x^2+1)^{-\frac{1}{2}}2x = -\frac{x}{\sqrt{x^2+1}}$

2. $f'(x) = \frac{2(x-1)(x+1)^2-(x-1)^2 2(x+1)}{(x+1)^2} = \frac{2(x-1)(x+1)((x+1)-(x-1))}{(x+1)^2} = \frac{4(x-1)}{x+1}$

3.

$$\begin{aligned}
f'(x) &= \frac{1}{2}\frac{1}{\sqrt{x+\sqrt{x+\sqrt{x}}}}(x+\sqrt{x+\sqrt{x}})' \\
&= \frac{1}{2}\frac{1}{\sqrt{x+\sqrt{x+\sqrt{x}}}}(1+\frac{1}{2}\frac{1}{\sqrt{x+\sqrt{x}}}(x+\sqrt{x})') \\
&= \frac{1}{2}\frac{1}{\sqrt{x+\sqrt{x+\sqrt{x}}}}(1+\frac{1}{2}\frac{1}{\sqrt{x+\sqrt{x}}}(1+\frac{1}{2}\frac{1}{\sqrt{x}}))
\end{aligned}$$

4.

$$\begin{aligned}
g'(t) &= \frac{(t^2)'(t-1)-t^2(t-1)'}{(t-1)^2} \\
&= \frac{2t(t-1)-t^2}{(t-1)^2} \\
&= \frac{t^2-2t}{(t-1)^2}
\end{aligned}$$

$$
\begin{aligned}
g''(t) &= \frac{(t^2-2t)'(t-1)^2-(t^2-2t)((t-1)^2)'}{(t-1)^4} \\
&= \frac{(2t-2)(t-1)^2-(t^2-2t)2(t-1)}{(t-1)^4} \\
&= \frac{2(t-1)^3-2t(t-1)(t-2)}{(t-1)^4} \\
&= \frac{2(t-1)((t-1)^2-t(t-2))}{(t-1)^4} \\
&= \frac{2(t^2-2t+1-t^2+2t)}{(t-1)^3} \\
&= \frac{2}{(t-1)^3}
\end{aligned}
$$

So $g''(2) = 2$.

5. $f'(x) = 3x^2 - 4x + 3$, so $f''(x) = 6x - 4$. The function is convex if and only if $f''(x) = 6x - 4 \geq 0$, so $x \geq \frac{2}{3}$. We conclude the function is concave on $[\frac{2}{3}, +\infty)$

6. $u'(y) = -A_2(-2y^{-3}) = 2A_2y^{-3}$
   $u''(y) = 2A_2(-3y^{-4}) = -6A_2y^{-4}$
   So $R = -\frac{yu''(y)}{u'(y)} = -\frac{y(-6A_2y^{-4})}{2A_2y^{-3}} = 3$

Second-Half

1. $f'(x) = e^{x^3+2x^2}(x^3+2x^2)' = e^{x^3+2x^2}(3x^2+4x)$

2. $f'(x) = \frac{1}{e^{2x}+1}(e^{2x}+1)' = \frac{2e^{2x}}{e^{2x}+1}$

3. $f(x) = (3^{x+1})' \ln(x-1) + 3^{x+1}(\ln(x-1))' = 3^{x+1}\ln 3 \ln(x-1) + \frac{3^{x+1}}{x-1}$

4. $f'(x) = (x^4)'e^{-2x} + x^4(e^{-2x})' = 4x^3e^{-2x} - 2x^4e^{-2x}$

5. Let $f'(x) = -2xe^{-x^2} \leq 0$, since $e^{-x^2} > 0$, we need $-2x \leq 0$, so $x \geq 0$.
   Let $f''(x) = -2(e^{-x^2} + x(-2xe^{-x^2})) = -2(1-2x^2)e^{-x^2} \geq 0$, so we need $1 - 2x^2 \leq 0$, we get $x^2 \geq \frac{1}{2}$, i.e. $x \geq \frac{\sqrt{2}}{2}$ or $x \leq -\frac{\sqrt{2}}{2}$.

Taking the intersection of the solutions, we conclude $x \geq \frac{\sqrt{2}}{2}$. So the function is decreasing and convex on $[\frac{\sqrt{2}}{2}, +\infty)$

6.
$$\ln f(x) = \ln(\frac{x+1}{x-1})^{\frac{1}{3}} = \frac{1}{3}\ln\frac{x+1}{x-1} = \frac{1}{3}(\ln(x+1) - \ln(x-1))$$

So if we take the derivative on both sides, we get

$$\frac{f'(x)}{f(x)} = \frac{1}{3}(\frac{1}{x+1} - \frac{1}{x-1})$$

7.
$$\ln f(x) = \ln(x^2+1)^x = x\ln(x^2+1)$$

Taking derivative on both sides, we get

$$\frac{f'(x)}{f(x)} = (x\ln(x^2+1))' = \ln(x^2+1) + x(\frac{2x}{x^2+1}) = \ln(x^2+1) + \frac{2x^2}{x^2+1}$$

So

$$f'(x) = f(x)(\ln(x^2+1) + \frac{2x^2}{x^2+1}) = (x^2+1)^x(\ln(x^2+1) + \frac{2x^2}{x^2+1})$$

8. (a). We regard y as a function of x, and use the implicit differetiation on the equation $2xy - 3y^2 = 9$, we get

$$2(y + xy') - 3(2yy') = 0$$

So $y' = \frac{y}{3y-x}$. At $(6,1)$, $y' = \frac{1}{3\times 1 - 6} = -\frac{1}{3}$.
The equation of the corresponding tangent line is

$$y - 1 = -\frac{1}{3}(x-6)$$

(b). In (a) we got
$$2(y + xy') - 3(2yy') = 0$$
dividing by 2:
$$y + xy' - 3yy' = 0$$

Differentiate both sides again with respect to x, we get:

$$y' + (y' + xy'') - 3(y'y' + yy'') = 0$$

Solving for y'':

$$y'' = \frac{(2-3y')y'}{3y-x} = \frac{2-3(\frac{y}{3y-x})}{3y-x}\frac{y}{3y-x} = \frac{(2(3y-x)-3y)y}{(3y-x)^3} = \frac{(3y-2x)y}{(3y-x)^3}$$

9. (a). $f'(x) = 5x^4 + 3x^2 + 2 > 0$, so f is strictly increasing on $\mathbb{R}$, so it has inverse function.

   (b). Observe that $f(0) = 1$, so $g'(1) = g'(f(0)) = \frac{1}{f'(0)} = \frac{1}{2}$