Homework I Solution

First-Half

1. (a). Since $x^2 - 1$ is the denominator, $x^2 - 1 \neq 0$, i.e. $x \neq \pm 1$, so the domain is $\{x \in \mathbb{R} : x \neq \pm 1\}$.

   (b). $f(-x) = \frac{-x}{(-x)^2-1} = -\frac{x}{x^2-1} = -f(x)$

2. (a). $x+2$ is under square root, so $x+2 \geq 0$, i.e. $x \geq -2$, so the domain is $[-2, +\infty)$.

   $\sqrt{x+2} \geq 0 \implies -\sqrt{x-2} \leq 0 \implies f(x) = 1 - \sqrt{x+2} \leq 1$, so the range is $(-\infty, 1]$.

   (b). $x - 2$ is the denominator, so $x - 2 \neq 0$, i.e. $x \neq 2$, so the domain is $\{x \in \mathbb{R} : x \neq 2\}$.

   $\frac{x-1}{x-2} = \frac{x-2+1}{x-2} = 1 + \frac{1}{x-2}$. $\frac{1}{x-2} \neq 0$, so $1 + \frac{1}{x-2} \neq 1$, so the range is $\{x \in \mathbb{R} : x \neq 1\}$.

3. (a). If $f(x_1) = f(x_2)$, then $\frac{1}{x_1+1} = \frac{1}{x_2+1}$, taking inverse of both sides, we get $x_1 + 1 = x_2 + 1$. Subtract 1 on both sides, we get $x_1 = x_2$, so the function is one-to-one.

   (b). $y = f(x) = \frac{1}{x+1}$, to $x + 1 = \frac{1}{y}$, so $x = \frac{1}{y} - 1$, we get the inverse function is $g(y) = \frac{1}{y} - 1$.

   The domain of f is $\{x \in \mathbb{R} : x \neq -1\}$, and the range of f is $\{x \in \mathbb{R} : x \neq 0\}$, so the domain of g is $\{x \in \mathbb{R} : x \neq 0\}$ and the range of g is $\{x \in \mathbb{R} : x \neq -1\}$.

4. f^{-1} tells you how much money you need when you buy a specified kilograms of carrots.

5. For any $x_1 < x_2$,

$$
\begin{aligned}
f(x_1) - f(x_2) &= (x_1^3 + 2x_1) - (x_2^3 + 2x_2) \\
&= (x_1^3 - x_2^3) + 2(x_1 - x_2) \\
&= (x_1 - x_2)(x_1^2 + x_1x_2 + x_2^2) + 2(x_1 - x_2) \\
&= (x_1 - x_2)(x_1^2 + x_1x_2 + x_2^2 + 2) \\
&= (x_1 - x_2)[(x_1 + \frac{x_2}{2})^2 + \frac{3}{4}x_2^2 + 2]
\end{aligned}
$$

Since $x_1 < x_2$, $x_1 - x_2 < 0$. Since $(x_1 + \frac{x_2}{2})^2 + \frac{3}{4}x_2^2 \geq 0$, $(x_1 + \frac{x_2}{2})^2 + \frac{3}{4}x_2^2 + 2 > 0$, so the product $(x_1 - x_2)[(x_1 + \frac{x_2}{2})^2 + \frac{3}{4}x_2^2 + 2] < 0$, i.e. $f(x_1) - f(x_2) < 0$, so $f(x_1) < f(x_2)$, we conclude f is a strictly increasing function.

6.

$$
\begin{aligned}
\lim_{x \to 3} \frac{x^2 - 2x - 15}{x - 5} &= \frac{\lim_{x \to 3} x^2 - 2x - 15}{\lim_{x \to 3} x - 5} \\
&= \frac{3^2 - 2 \times 3 - 15}{3 - 5} \\
&= \frac{-12}{-2} \\
&= 6
\end{aligned}
$$

7.

$$
\begin{aligned}
\lim_{x \to 5} \frac{x^2 - 2x - 15}{x - 5} &= \lim_{x \to 5} \frac{(x-5)(x+3)}{x - 5} \\
&= \lim_{x \to 5} x + 3 \\
&= 5 + 3 \\
&= 8
\end{aligned}
$$

8.

$$\begin{aligned}
\lim_{x\to+\infty}(\sqrt{x+1}-\sqrt{x}) &= \lim_{x\to+\infty}(\sqrt{x+1}-\sqrt{x})\frac{\sqrt{x+1}+\sqrt{x}}{\sqrt{x+1}+\sqrt{x}} \\
&= \lim_{x\to+\infty}\frac{(\sqrt{x+1})^2-(\sqrt{x})^2}{\sqrt{x+1}+\sqrt{x}} \\
&= \lim_{x\to+\infty}\frac{1}{\sqrt{x+1}+\sqrt{x}} \\
&= 0
\end{aligned}$$

Second-Half

1. (a).

$$\begin{aligned}
f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} &= \lim_{h\to 0}\frac{[(x+h)^2-3(x+h)+1]-[x^2-3x+1]}{h} \\
&= \lim_{h\to 0}\frac{[(x+h)^2-x^2]-3[(x+h)-3x]}{h} \\
&= \lim_{h\to 0}\frac{2xh+h^2-3h}{h} \\
&= \lim_{h\to 0} 2x+h-3 \\
&= 2x-3
\end{aligned}$$

(b). $f'(x)=(x^2-3x+1)'=(x^2)'-3(x)'=2x-3$

2. (a).

$$\begin{aligned}
f'(x) &= (x^{-\frac{2}{3}}-3)'(x^{\frac{1}{2}}+1)+(x^{-\frac{2}{3}}-3)(x^{\frac{1}{2}}+1)' \\
&= -\frac{2}{3}x^{-\frac{5}{3}}(x^{\frac{1}{2}}+1)+(x^{-\frac{2}{3}}-3)\frac{1}{2}x^{-\frac{1}{2}} \\
&= -\frac{2}{3}x^{-\frac{7}{6}}-\frac{2}{3}x^{-\frac{5}{3}}+\frac{1}{2}x^{-\frac{7}{6}}-\frac{3}{2}x^{-\frac{1}{2}} \\
&= -\frac{1}{6}x^{-\frac{7}{6}}-\frac{2}{3}x^{-\frac{5}{3}}-\frac{3}{2}x^{-\frac{1}{2}}
\end{aligned}$$

(b).

$$\begin{aligned}
f'(x) &= \frac{(3x)'(x^2+2)-(3x)(x^2+2)'}{(x^2+2)^2} \\
&= \frac{3(x^2+2)-3x(2x)}{(x^2+2)^2} \\
&= \frac{-3x^2+6}{(x^2+2)^2}
\end{aligned}$$

(c).

$$\begin{aligned}
f'(x) &= \frac{(x^4+1)'(x^2+1)(x+3)-(x^4+1)[(x^2+1)(x+3)]'}{(x^2+1)^2(x+3)^2} \\
&= \frac{4x^3(x^2+1)(x+3)-(x^4+1)[2x(x+3)+(x^2+1)]}{(x^2+1)^2(x+3)^2} \\
&= \frac{4x^3(x^2+1)(x+3)-(x^4+1)[3x^2+6x+1]}{(x^2+1)^2(x+3)^2} \\
&= \frac{x^6+6x^5+3x^4+12x^3-3x^2-6x-1}{(x^2+1)^2(x+3)^2}
\end{aligned}$$

3. $f(1) = 2$

$$\begin{aligned}
f'(x) &= \frac{[(x^3+1)(x+1)]'(x^2+1)-(x^3+1)(x+1)(x^2+1)'}{(x^2+1)^2} \\
&= \frac{(4x^3+3x^2+1)(x^2+1)-(x^3+1)(x+1)2x}{(x^2+1)^2}
\end{aligned}$$

So $f'(1) = 2$

So the equation for the tangent line at $x = 1$ is

$$y - 2 = 2(x-1)$$

4. Let $f'(x) = 3x^2 - 12x + 9 = 3(x-1)(x-3) \geq 0$, we see it holds if and only if

$$\begin{cases} x-1 \geq 0 \\ x-3 \geq 0 \end{cases}$$

or

$$\begin{cases} x - 1 \leq 0 \\ x - 3 \leq 0 \end{cases}$$

if and only if $x \geq 3$ or $x \leq 1$, so the function is increasing on $(-\infty, 1]$ and $[3, +\infty)$.

5. $C(x+h) - C(x) \approx C'(x)h$, so when $x = 10,000$, $h = 2$,

$$C(1,000 + 2) - C(1,000) = C'(1,000) \times 2 = 15 \times 2 = 30$$

So there will be 30 dollars extra cost per day.