1. $f(x) = \frac{1}{(\sqrt{x})^2+1}$. What's the domain and range of f?

Since x is under squareroot, $x \geq 0$.

$\sqrt{x} \geq 0 \Rightarrow (\sqrt{x})^2 \geq 0 \Rightarrow (\sqrt{x})^2+1 \geq 1$, so the denominator is never 0

we conclude the domain is $[0, +\infty)$

$(\sqrt{x})^2+1 \geq 1 \Rightarrow 0 < \frac{1}{(\sqrt{x})^2+1} \leq 1$.

so the range is $(0, 1]$

2. $f(x) = \frac{1}{2x-1}$. Compute the inverse function g of f, and also find the domain and range of g.

$y = f(x) = \frac{1}{2x-1} \Rightarrow 2x-1 = \frac{1}{y} \Rightarrow 2x = \frac{1}{y}+1$

$\Rightarrow x = \frac{1}{2}(\frac{1}{y}+1)$

So $x = g(y) = \frac{1}{2}(\frac{1}{y}+1)$

The domain of $f(x)$ is $2x-1 \neq 0 \Rightarrow x \neq \frac{1}{2}$.

So the range of $f(x)$ is $(-\infty, \frac{1}{2}) \cup (\frac{1}{2}, +\infty)$

The range of $f(x)$ is $(-\infty, 0) \cup (0, +\infty)$

So the domain of $f(x)$ is $(-\infty, 0) \cup (0, +\infty)$

3. Compute the limit

$$\lim_{x\to -4} \frac{x^2+9x+20}{x^2-16}$$

$$\lim_{x\to -4} \frac{x^2+9x+20}{x^2-16} = \lim_{x\to -4} \frac{(x+5)(x+4)}{(x-4)(x+4)}$$

$$= \lim_{x\to -4} \frac{x+5}{x-4}$$

$$= \frac{-4+5}{-4-4}$$

$$= -\frac{1}{8}$$

4. Compute the derivative of $f(x) = 3x^2$ using **definition** $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$, without applying the rules for differentiation.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{3(x+h)^2 - 3x^2}{h}$$

$$= \lim_{h \to 0} \frac{3[(x+h)^2 - x^2]}{h}$$

$$= \lim_{h \to 0} \frac{3[x^2 + 2hx + h^2 - x^2]}{h}$$

$$= \lim_{h \to 0} \frac{3 \cdot (2hx + h^2)}{h}$$

$$= \lim_{h \to 0} 3 \cdot (2x + h)$$

$$= 6x$$

5. Compute the derivative of $f(x) = (x+1)e^{x^3-1}$

$$f'(x) = (x+1)'e^{x^3-1} + (x+1)(e^{x^3-1})'$$

$$= e^{x^3-1} + (x+1)\cdot e^{x^3-1}\cdot 3x^2$$

$$= [1 + (x+1)\cdot 3x^2]e^{x^3-1}$$

$$= [3x^3 + 3x^2 + 1]e^{x^3-1}$$

6. Compute the derivative of $f(x) = \ln(e^x + e^{-x})$

$$f'(x) = \frac{1}{e^x + e^{-x}} \cdot (e^x + e^{-x})'$$

$$= \frac{1}{e^x + e^{-x}} (e^x - e^{-x})$$

$$= \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

7. Compute the derivative of $f(x) = \frac{(x+1)^2}{x-3}$.

$$f'(x) = \frac{2(x+1)(x-3) - (x+1)^2 \cdot 1}{(x-3)^2}$$

$$= \frac{2(x^2-2x-3) - (x^2+2x+1)}{(x-3)^2}$$

$$= \frac{x^2-6x-7}{(x-3)^2}$$

$$= \frac{(x-7)(x+1)}{(x-3)^2}$$

8. $f(x) = x^3 - 3x^2 - 9x + 157$. Find all the intervals on which f is decreasing and convex.

$f'(x) = 3x^2 - 6x - 9 = 3(x^2 - 2x - 3) = 3(x-3)(x+1) \leq 0$

so $-1 \leq x \leq 3$

$f''(x) = 6x - 6 = 6(x-1) \geqslant 0$

so $x \geqslant 1$

Taking the intersection of the two parts. we conclude $1 \leq x \leq 3$

so $[1, 3]$.

9. Find the equation of the tangent line for the curve $2x^2 + 6xy + y^2 = 18$ at the point $(x, y) = (1, 2)$.

We take implicit differentiation

$$2 \cdot 2x + 6(y + xy') + 2y \cdot y' = 0$$

$$4x + 6y + 6xy' + 2yy' = 0$$

$$y' = \frac{-4x - 6y}{6x + 2y} = \frac{-4 \times 1 - 6 \times 2}{6 \times 1 + 2 \times 2} = -\frac{8}{5}$$

So the equation of the tangent line is

$$y - 2 = -\frac{8}{5}(x - 1)$$

10. $f(x) = x^7 + x^3 + x + 1$, and g is the inverse function of f. Compute $g'(1)$ and $g'(4)$.

$$f'(x) = 7x^6 + 3x^2 + 1$$

$$1 = f(0) \quad \text{and} \quad 4 = f(1)$$

So $$g'(1) = g'(f(0)) = \frac{1}{f'(0)} = 1$$

$$g'(4) = g'(f(1)) = \frac{1}{f'(1)} = \frac{1}{11}$$

11. Find the derivative of $f(x) = (x+1)^{x-1}$

$$\ln f(x) = \ln (x+1)^{x-1} = (x-1)\ln(x+1)$$

Take derivative on both sides.

$$\frac{f'(x)}{f(x)} = \ln(x+1) + (x-1)\cdot\frac{1}{x+1}$$

$$= \ln(x+1) + \frac{x-1}{x+1}$$

So $f'(x) = \left(\ln(x+1) + \frac{x-1}{x+1}\right)\cdot f(x)$

$$= \left(\ln(x+1) + \frac{x-1}{x+1}\right)\cdot(x+1)^{x-1}$$

12. $C(x)$ is the cost of producing x units of commodities,

$$C(x) = 900x^{\frac{1}{3}} + 5000$$

(a). what's the marginal cost at $x = 1000$?

(b). Use this marginal cost to approximate the increase in cost if the production is increased from 1000 units to 1002 units.

(a) $C'(x) = 900 \cdot \frac{1}{3} x^{-\frac{2}{3}} = 300 \cdot x^{-\frac{2}{3}}$

So $C'(1000) = 300 \cdot (1000)^{-\frac{2}{3}} = 300 \cdot (10^3)^{-\frac{2}{3}}$

$= 300 \cdot 10^{-2}$

$= 3$

(b) $C(1002) - C(1000) \approx C'(1000) \times (1002 - 1000) = 3 \times 2 = 6$