Exponential Functions, Logarithmic Functions

An exponential function is a function of the form $f(x) = a^x$, where $a > 0$, $a \neq 1$ is a constant

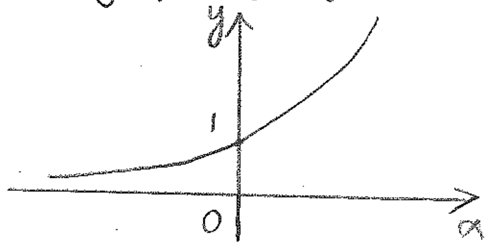
Domain of $f(x) = a^x$ is $\mathbb{R}$, and range of $f(x) = a^x$ is $(0, +\infty)$.

Theorem. $f(x) = a^x$ has the following properties:

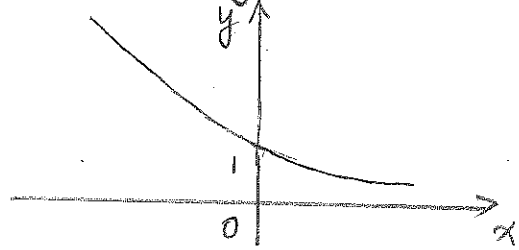
① $f(0) = 1$, $f(1) = a$.

② $f(x_1 + x_2) = a^{x_1 + x_2} = a^{x_1} \cdot a^{x_2} = f(x_1) \cdot f(x_2)$

③ The graph of f looks like the following:



if $a > 1$



if $0 < a < 1$

Now let's study the derivative of $f(x) = a^x$.

By definition, $f'(x) = \lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \dfrac{a^{x+h} - a^x}{h} = a^x \lim_{h \to 0} \dfrac{a^h - 1}{h}$

i.e. $(a^x)' = a^x \cdot \lim_{h \to 0} \dfrac{a^h - 1}{h}$.

We define $e \in \mathbb{R}$ to be the real number satisfying $\lim_{h \to 0} \dfrac{e^h - 1}{h} = 1$

Then $(e^x)' = e^x$. By some more advuced mathematics, people have proved e is an irrational number and $e \approx 2.71828\ldots$

The function $f(x) = e^x$ is called the natural exponential function.

The logarithmic function $g(y) = \log_a(y)$ $(a>0, a\neq 1)$ is the inverse function to the exponential function $f(x) = a^x$.

(Recall that $a^x = y \Leftrightarrow x = \log_a y$)

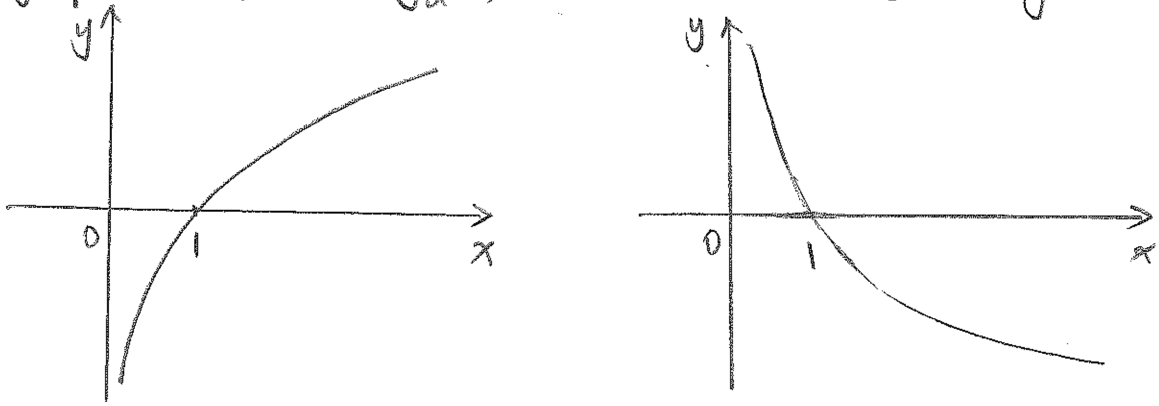
So for each $a>0$, $a\neq 1$. the logarithmic function $g(x) = \log_a(x)$ has the following properties:

① Domain of $g(x)$ is $(0, +\infty)$, range of $g(x)$ is $\mathbb{R}$

② $g(1) = \log_a 1 = 0$. $g(a) = \log_a a = 1$

③ $g(x_1 x_2) = \log_a(x_1 x_2) = \log_a x_1 + \log_a x_2$

④ The graph of $g(x) = \log_a(x)$ looks like the following:



$a>1$ $\qquad\qquad$ $0<a<1$

When we take $a=e$, $g(x) = \log_e(x)$. We call it the natural logarithmic function, and write it as $g(x) = \ln(x)$

Theorem. $(\ln(x))' = \frac{1}{x}$.

Proof. $x = e^{\ln x}$. take derivative on both sides.

$$1 = e^{\ln x} \cdot (\ln x)'$$

$$1 = x \cdot (\ln x)'$$

So $(\ln x)' = \frac{1}{x}$

Now we can find the derivative of $f(x) = a^x$ and $g(x) = \log_a x$ for any $a > 0$, $a \neq 1$:

$$(a^x)' = [(e^{\ln a})^x]' = (e^{x \ln a})' = e^{x \ln a} \cdot \ln a = (e^{\ln a})^x \cdot \ln a = a^x \ln a$$

$$(\log_a x)' = \left(\frac{\ln x}{\ln a}\right)' = \frac{1}{\ln a} (\ln x)' = \frac{1}{x \ln a}$$

apply the change of base formula $\boxed{\log_a b = \frac{\log_c b}{\log_c a}}$

Example. $f(x) = \pi^x$, then $f'(x) = \pi^x \cdot \ln \pi$

$g(x) = \log_\pi x$, then $g'(x) = \frac{1}{x \ln \pi}$

Example. $f(x) = e^{x^2+x+1}$

$$f'(x) = e^{x^2+x+1} \cdot (2x+1)$$

Example. $f(x) = \ln(x^2+x+1)$

$$f'(x) = \frac{1}{x^2+x+1} \cdot (2x+1) = \frac{2x+1}{x^2+x+1}$$

Example. $f(x) = \ln(e^{-x^2}+1)$

$$= \frac{1}{e^{-x^2}+1} \cdot (e^{-x^2}+1)'$$

$$= \frac{1}{e^{-x^2}+1} \cdot e^{-x^2} \cdot (-2x)$$

$$= \frac{-2x e^{-x^2}}{e^{-x^2}+1}$$

Example. $f(x) = x^n e^x$

$$f'(x) = (x^n)'e^x + x^n(e^x)' = nx^{n-1}e^x + x^n e^x = (n+x)x^{n-1}e^x$$

Example $f(x) = x^x$, $(x>0)$

Write $f(x) = x^x = (e^{\ln x})^x = e^{x\ln x}$

so $f'(x) = (e^{x\ln x})' = e^{x\ln x} \cdot (\ln x + x \cdot \frac{1}{x})$

$$= x^x \cdot (\ln x + 1)$$

Example. If f is a function with range contained in $(0, +\infty)$.

Then let $g(x) = \ln f(x)$.

$g'(x) = (\ln f(x))' = \frac{f'(x)}{f(x)}$, which gives the relative rate of change of $f(x)$ at x.

If $F(x) = f(x)\,g(x)$, $\ln F(x) = \ln f(x)g(x) = \ln f(x) + \ln g(x)$

So $(\ln F(x))' = (\ln f(x))' + (\ln g(x))'$

$$\frac{F'(x)}{F(x)} = \frac{f'(x)}{f(x)} + \frac{g'(x)}{g(x)}$$

this is another proof of the above equality.

(recall we have proved it before using product rule)

The method used in the last example is called logarithmic differentiation, i.e. Let $g(x) = \ln f(x)$. then $g'(x) = \frac{f'(x)}{f(x)}$