Chain Rule

Composition of Functions:

If y is a function of u. $y=f(u)$, and

u is a function of x, $u=g(x)$

Then y is a composition function of x: $y=f(g(x))$

Notation: We also write $f\circ g$ to represent the map above which sends x to y, and we call it the composition of f with g.

Example. $f(u)=u^2+2u$ $\quad g(x)=x+1$.

$$f\circ g(x)=f(g(x))=f(x+1)=(x+1)^2+2(x+1)$$

The Chain Rule:

If g is differentiable at x_0 and f is differentiable at $u_0=g(x_0)$

Then $F(x)=f(g(x))$ is differentiable at x_0, and:

$$F'(x_0)=f'(u_0)g'(x_0)=f'(g(x_0))\,g'(x_0)$$

Proof. Let $\varphi(u)=\begin{cases}\dfrac{f(u)-f(u_0)}{u-u_0} & \text{if } u\neq u_0\\ f'(u_0) & \text{if } u=u_0\end{cases}$

$$\psi(x)=\begin{cases}\dfrac{g(x)-g(x_0)}{x-x_0} & \text{if } x\neq x_0\\ g'(x_0) & \text{if } x=x_0\end{cases}$$

Note that $\displaystyle\lim_{u\to u_0}\varphi(u)=\lim_{u\to u_0}\frac{f(u)-f(u_0)}{u-u_0}=f'(u_0)$

$$\lim_{x\to x_0}\psi(x)=\lim_{x\to x_0}\frac{g(x)-g(x_0)}{x-x_0}=g'(x_0)$$

Also $\begin{cases} f(u) - f(u_0) = \varphi(u)(u - u_0) \\ g(x) - g(x_0) = \psi(x)(x - x_0) \end{cases}$

So $F(x_0+h) - F(x_0) = f(g(x_0+h)) - f(g(x_0))$

$$= \varphi(g(x_0+h))(g(x_0+h) - g(x_0))$$

$$= \varphi(g(x_0+h)) \cdot \psi(x_0+h)(x_0+h-x_0)$$

$$= \varphi(g(x_0+h))\,\psi(x_0+h)\,h$$

So $F'(x_0) = \lim_{h\to 0} \frac{F(x_0+h) - F(x_0)}{h} = \lim_{h\to 0} \varphi(g(x_0+h))\,\psi(x_0+h)$

$$= \varphi(g(x_0))\,\psi(x_0)$$

$$= f'(g(x_0)) \cdot g'(x_0)$$

Example. Compute the derivative of $f(x) = (x^2+2x)^9$

Observe $f(x) = g(k(x))$ where $g(u) = u^9$ and $u = k(x) = x^2+2x$

$$f'(x) = g'(k(x)) \cdot k'(x) = 9 \cdot (x^2+2x)^8 \cdot (2x+2)$$

$$= 18(x+1)(x^2+2x)^8$$

Example. Compute the derivative of $f(x) = \sqrt{x^3+2}$

$f(x) = g(k(x))$, where $g(u) = \sqrt{u}$, $u = k(x) = x^3+2$

$$f'(x) = g'(k(x)) \cdot k'(x) = \frac{1}{2} \cdot \frac{1}{\sqrt{x^3+2}} \cdot 3x^2 = \frac{3x^2}{2\sqrt{x^3+2}}$$

Example. Find the derivative function of $f(x) = \frac{(2x-1)^2}{x^2+1}$

$$f'(x) = \frac{[(2x-1)^2]'(x^2+1) - (2x-1)^2(x^2+1)'}{(x^2+1)^2}$$

$$= \frac{2(2x-1)\cdot 2\cdot(x^2+1) - (2x-1)^2\cdot 2x}{(x^2+1)^2}$$

$$= \frac{4(2x-1)(x^2+1) - 2x(2x-1)^2}{(x^2+1)^2}$$

$$= \frac{2(2x-1)(x+2)}{(x^2+1)^2}$$

Example. Find the derivative function of $f(x) = (x^2-1)^2(x-2)^3$

$$f'(x) = [(x^2-1)^2]'(x-2)^3 + (x^2-1)^2[(x-2)^3]'$$

$$= 2(x^2-1)\cdot 2x\cdot(x-2)^3 + (x^2-1)^2\cdot 3(x-2)^2\cdot 1$$

$$= 4x(x^2-1)(x-2)^3 + 3(x^2-1)^2(x-2)^2$$

$$= (x^2-1)(x-2)^2(7x^2-8x-3)$$

Example. Find the derivative function of $f(x) = \frac{1}{(x^2+1)^2+\sqrt{x}}$

$$f'(x) = -\frac{1}{((x^2+1)^2+\sqrt{x})^2}\cdot((x^2+1)^2+\sqrt{x})'$$

$$= -\frac{2(x^2+2)\cdot 2x + \frac{1}{2}x^{-\frac{1}{2}}}{((x^2+1)^2+\sqrt{x})^2}$$

$$= -\frac{8x(x^2+2)\sqrt{x}+1}{2((x^2+1)^2+\sqrt{x})^2\cdot\sqrt{x}}$$

Higher Order Derivatives

F is a differentiable function with derivative f'. We call f' the first derivative. If f' is also differentiable, we use f'' to denote the derivative of f', and call it the second order derivative of f.

Example. $f(x) = 3x^2 + 6x + 5$

Then $f'(x) = 6x + 6$

$f''(x) = 6$

Example Consider a particle moving along a straight line. It's displacement is a function with respect to time : $S(t) = t^5 + 3t^2 - 2t$

Then it's velocity at time t is

$$v(t) = S'(t) = 5t^4 + 6t - 2$$

it's acceleration at time t is

$$a(t) = v'(t) = S''(t) = 20t^3 + 6$$

An application of the second derivative is to study the Convex and Concave Functions.

A first observation is that $f''(x)$ is the derivative of $f'(x)$, so

$f'(x)$ is increasing on an interval $I \iff f''(x) \geq 0$ on I

$f'(x)$ is decreasing on an interval $I \iff f''(x) \leq 0$ on I

Geometrically. $f'(x)$ is increasing $\iff$ slope of $f(x)$ is increasing

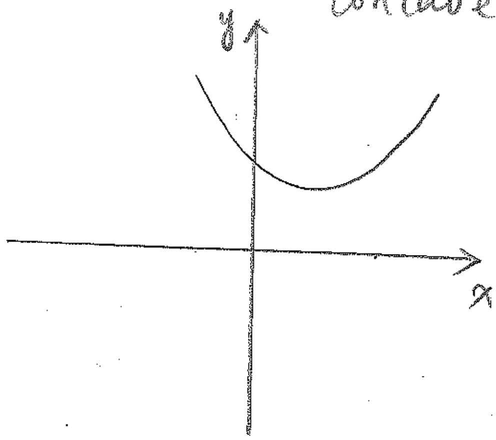
$f'(x)$ is decreasing $\iff$ slope of $f(x)$ is decreasing

Now we introduce the concept of convex functions and concave functions:

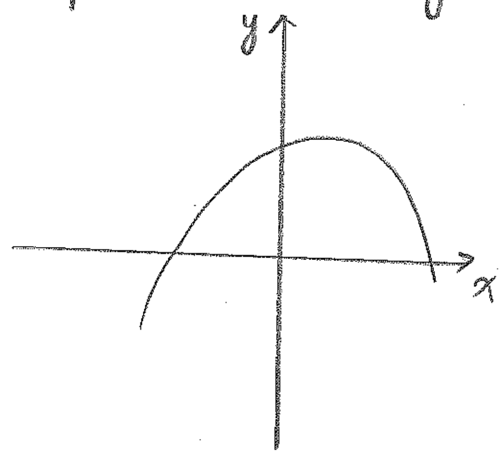
We define f is convex on $I \iff f''(x) \geq 0$ on I

f is concave on $I \iff f''(x) \leq 0$ on I.

So a function is convex if its slope is increasing, and concave if its slope is decreasing.



Convex



Concave

Example. Find all intervals on which $f(x) = x^3 - x^2$ is convex

$f'(x) = 3x^2 - 2x \qquad f''(x) = 6x - 2$

So $f''(x) = 6x - 2 \geq 0$ iff $x \geq \frac{1}{3}$.

We conclude $f(x)$ is convex on $[\frac{1}{3}, +\infty)$

Example. Find all intervals on which $f(x) = x^4 - 6x^2$ is decreasing and concave.

$f'(x) = 4x^3 - 12x$. $f''(x) = 12x^2 - 12$

$f'(x) = 4x(x^2-3) \leq 0 \iff x \leq -\sqrt{3}$ or $0 \leq x \leq \sqrt{3}$

So f is decreasing on $(-\infty, -\sqrt{3}]$ and $[0, \sqrt{3}]$

$f''(x) = 12x^2 - 12 = 12(x^2-1) \leq 0 \iff -1 \leq x \leq 1$.

So f is concave on $[-1, 1]$.

We conclude f is both decreasing and concave on $[0, 1]$.

We have seen the second derivative, which is to take derivative of a function two times. More generally, we can take derivative n times, if each time the derivative is still differentiable. The function we obtain after taking derivatives n times on f is called the n-th derivative of f. and denoted as $f^{(n)}(x)$. n is called the order of the derivative.

Example. $f(x) = x^4 + x^3 + x^2 + x + 1$

$$f'(x) = 4x^3 + 3x^2 + 2x + 1$$

$$f''(x) = 12x^2 + 6x + 2$$

$$f^{(3)}(x) = 24x + 6$$

$$f^{(4)}(x) = 24$$

$$f^{(n)}(x) = 0 \quad \text{for } n \geq 5$$