Basic Differentiation

We have defined the derivative of a function. Differentiation is the process of finding the derivative of a function. We can find the derivative by definition, which is direct, but often hard. We are going to study some rules of differentiation, with whose help the computation is much easier.

Theorem ① If $f(x) \equiv C$ (C is a real number) is a constant function then $f'(x) \equiv 0$.

② (Power Rule) $f(x) = x^a$ ($a \neq 0$, constant), then $f'(x) = a x^{a-1}$

(In particular, we have proved the case when a is a positive integer)

Theorem If f is a function and A is a constant, then:

① $g(x) = f(x) + A \Rightarrow g'(x) = f'(x)$

② $g(x) = Af(x) \Rightarrow g'(x) = A f'(x)$

Proof. ① $g'(x) = \lim_{h \to 0} \frac{g(x+h) - g(x)}{h} = \lim_{h \to 0} \frac{(f(x+h) + A) - (f(x) + A)}{h}$

$$= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

② $g'(x) = \lim_{h \to 0} \frac{g(x+h) - g(x)}{h}$

$$= \lim_{h \to 0} \frac{Af(x+h) - Af(x)}{h}$$

$$= \lim_{h \to 0} A \cdot \frac{f(x+h) - f(x)}{h} = A \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = Af'(x)$$

If the derivative of a function f exists at x, we say f is differentiable at x.

<u>Theorem</u>: If f & g are differentiable at x, then $f+g$, $f-g$, $f\cdot g$ are differentiable at x. If f & g are differentiable at x and $g(x)\neq 0$, then $\frac{f}{g}$ is differentiable at x. There are rules to compute the derivative of $f\pm g$, $f\cdot g$ and $\frac{f}{g}$:

① $F(x) = f(x) \pm g(x) \Rightarrow F'(x) = f'(x) \pm g'(x)$

② $F(x) = f(x)\cdot g(x) \Rightarrow F'(x) = f'(x)g(x) + f(x)g'(x)$

③ $F(x) = \dfrac{f(x)}{g(x)} \quad (g(x)\neq 0) \Rightarrow F'(x) = \dfrac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}$

Proof ① We will do the case $F(x) = f(x)+g(x)$, the other case can be proved similarly.

$$F'(x) = \lim_{h\to 0}\frac{F(x+h)-F(x)}{h} = \lim_{h\to 0}\frac{(f(x+h)+g(x+h))-(f(x)+g(x))}{h}$$

$$= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} + \lim_{h\to 0}\frac{g(x+h)-g(x)}{h}$$

$$= f'(x)+g'(x)$$

② $$F'(x) = \lim_{h\to 0}\frac{F(x+h)-F(x)}{h} = \lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x)g(x)}{h}$$

$$= \lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x)g(x+h)+f(x)g(x+h)-f(x)g(x)}{h}$$

$$= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\,g(x+h) + \lim_{h\to 0} f(x)\,\frac{g(x+h)-g(x)}{h}$$

$$= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\,\lim_{h\to 0} g(x+h) + f(x)\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}$$

$$= f'(x)g(x) + f(x)g'(x)$$

③. We first compute the derivative of $k(x) = \frac{1}{g(x)}$.

$$k'(x) = \lim_{h\to 0} \frac{k(x+h)-k(x)}{h} = \lim_{h\to 0} \frac{\frac{1}{g(x+h)} - \frac{1}{g(x)}}{h}$$

$$= \lim_{h\to 0} \frac{g(x)-g(x+h)}{h\, g(x+h) g(x)}$$

$$= -\frac{1}{g(x)} \lim_{h\to 0} \frac{1}{g(x+h)} \lim_{h\to 0} \frac{g(x+h)-g(x)}{h}$$

$$= -\frac{1}{g(x)^2} \cdot g'(x)$$

$$= -\frac{g'(x)}{g(x)^2}$$

Next $F(x) = \frac{f(x)}{g(x)} = f(x)k(x)$, by Part ②. we have:

$$F'(x) = (f(x)k(x))' = f'(x)k(x) + f(x)k'(x)$$

$$= f'(x)\frac{1}{g(x)} + f(x)\cdot \frac{-g'(x)}{g(x)^2}$$

$$= \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}$$

Now with all the theorems we have learned so far, we can compute the derivative of much complicated functions, and this will in general be a much faster way than computing the derivative by definition.

Example. Find the derivative of $f(x) = 3x^5 + 2x^2 + 1$

$$f'(x) = (3x^5)' + (2x^2)' + 1'$$

$$= 3(x^5)' + 2(x^2)' + 0$$

$$= 3\cdot 5x^4 + 2\cdot 2x^1$$

$$= 15x^4 + 4x$$

Example Find the derivative of $f(x) = \sqrt{x} + \sqrt[3]{x^2}$

$$f(x) = \sqrt{x} + \sqrt[3]{x^2} = x^{\frac{1}{2}} + x^{\frac{2}{3}}$$

so $$f'(x) = (x^{\frac{1}{2}})' + (x^{\frac{2}{3}})' = \frac{1}{2}x^{-\frac{1}{2}} + \frac{2}{3}x^{-\frac{1}{3}} = \frac{1}{2\sqrt{x}} + \frac{2}{3\cdot\sqrt[3]{x}}$$

Example Find the derivative of $f(x) = \dfrac{\sqrt{x}-1}{x^2+3}$

$$f'(x) = \frac{(\sqrt{x}-1)'(x^2+3) - (\sqrt{x}-1)(x^2+3)'}{(x^2+3)^2} = \frac{(x^{\frac{1}{2}})'(x^2+3) - (x^{\frac{1}{2}}-1)(x^2)'}{(x^2+3)^2}$$

$$= \frac{\frac{1}{2}x^{-\frac{1}{2}}(x^2+3) - (x^{\frac{1}{2}}-1)\cdot 2x}{(x^2+3)^2}$$

$$= \frac{(x^2+3) - (x^{\frac{1}{2}}-1)2x\cdot 2x^{\frac{1}{2}}}{2x^{\frac{1}{2}}(x^2+3)^2}$$

$$= \frac{-3x^2 + 4x\sqrt{x} + 3}{2\sqrt{x}(x^2+3)^2}$$

Example Show that if $F(x) = f(x)g(x)$, $G(x) = \dfrac{f(x)}{g(x)}$ $(g(x) \neq 0)$ then

$$\frac{F'(x)}{F(x)} = \frac{f'(x)}{f(x)} + \frac{g'(x)}{g(x)}, \quad \frac{G'(x)}{G(x)} = \frac{f'(x)}{f(x)} - \frac{g'(x)}{g(x)}$$

$$\frac{F'(x)}{F(x)} = \frac{f'(x)g(x) + f(x)g'(x)}{f(x)g(x)} = \frac{f'(x)}{f(x)} + \frac{g'(x)}{g(x)}$$

$$\frac{G'(x)}{G(x)} = \frac{\dfrac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}}{\dfrac{f(x)}{g(x)}} = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}\cdot\frac{g(x)}{f(x)}$$

$$= \frac{f'(x)g(x) - f(x)g'(x)}{f(x)g(x)}$$

$$= \frac{f'(x)}{f(x)} - \frac{g'(x)}{g(x)}$$

Derivative and Monotonicity

<u>Theorem</u>. • $f'(x) \geq 0$ for all x in the interval $I \Leftrightarrow f$ is increasing in I

• $f'(x) \leq 0$ for all x in the interval $I \Leftrightarrow f$ is decreasing in I

• $f'(x) = 0$ for all x in the interval $I \Leftrightarrow f$ is constant

• $f'(x) > 0$ for all x in the interval $I \Rightarrow f$ is strictly increasing in I

• $f'(x) < 0$ for all x in the interval $I \Rightarrow f$ is strictly decreasing in I

Remark f is strictly increasing in I does not necessarily imply $f'(x) > 0$ for all x in I

For example $f(x) = x^3$ is strictly increasing on $\mathbb{R}$

but $f'(0) = 0$

Example. Find all the intervals on which $f(x) = 2x^3 - 15x^2 + 36x + 99$ is increasing

$$f'(x) = 6x^2 - 30x + 36 = 6(x^2 - 5x + 6) = 6(x-2)(x-3) \geq 0$$

So $x \geq 3$ or $x \leq 2$

So $f(x)$ is increasing on $(-\infty, 2]$ and $[3, +\infty)$