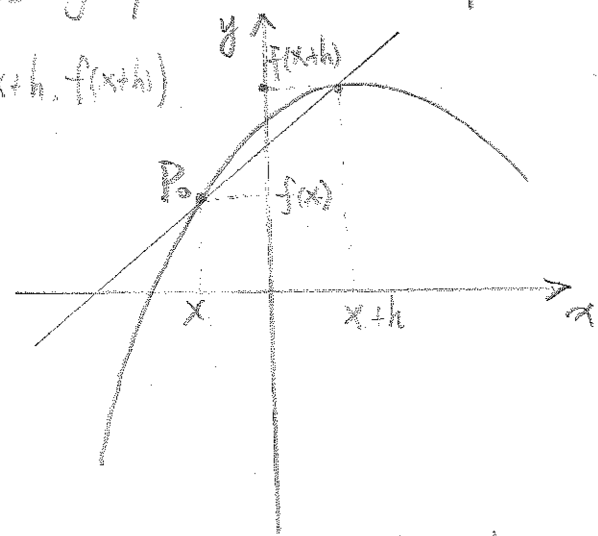
# Slope, Tangent & Derivatives

The graph of a function $f$ passes through a point $P_0 = (x, f(x))$. We want to describe how steep the graph is at the point $P_0$.




Take a point on the graph $P_h = (x+h, f(x+h))$ and connect $P$ and $P_h$ by a straight line $L_h$.
As $|h|$ is getting smaller and smaller, the straight line $L_h$ becomes better and better approximation of the graph near $P_0$. The slope of the graph at $P$ can be described as the limit case of the slope of $L_h$ as $h$ approaches $0$, and also note that the steepness of the straight line is given by the formula $\frac{f(x+h)-f(x)}{h}$, which is called the slope of the straight line.

Geometrically, as $h \to 0$ the straight line $L_h$ will tend to a straight line $L_0$ which intersects the graph at only one point near $P_0$. This straight line has the same slope as the graph at $P_0$, and the slope is the limit: $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$

So by the above observations we will make the following definitions:

- The derivative of a function $f$ at point $a$, denoted by $f'(a)$ is given by $f'(a) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$
- The tangent line of the graph $y=f(x)$ at point $(a, f(a))$ is:
  $y - f(a) = f'(a)(x-a)$

Remark If we use the language of the previous class, we can say that the derivative of a function $f$ at $a$ is the instantaneous rate of change of $f$ at $a$, and the slope of the straight line $L_h$ is the average rate of change of $f$ over the interval $[a, a+h]$

A main objective of this course is to learn how to compute derivative! Today we will use the most fundamental way: by definition.

How to compute the derivative by definition?

Step 1. compute $f(a+h)$

Step 2. compute $\frac{f(a+h)-f(a)}{h}$ (also try to simplify the expression)

Step 3. take the limit $\lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$

Example. $f(x)=x^2$. compute $f'(a)$

$$f(a+h)=(a+h)^2=a^2+2ah+h^2$$

$$\frac{f(a+h)-f(a)}{h}=\frac{a^2+2ah+h^2-a^2}{h}=\frac{2ah+h^2}{h}=2a+h$$

so $f'(a)=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}=\lim_{h\to 0}(2a+h)=2a$

Now we can define the derivative function $f'$ of a given function $f$ in the way that for each $x$ in the domain, we assign the real number $f'(x)$ to it. To compute the derivative function is the same way as to compute the derivative, just replace $a$ by $x$ in all the computations.

Example. Compute $f'(x)$ when $f(x) = x^n$, then find $f'(2)$

$$f(x+h) = (x+h)^n = x^n + \binom{n}{1}x^{n-1}h + \binom{n}{2}x^{n-2}h^2 + \cdots + \binom{n}{n-2}x^2h^{n-2} + \binom{n}{n-1}xh^{n-1} + h^n$$

$$\frac{f(x+h)-f(x)}{h} = \frac{\binom{n}{1}x^{n-1}h + \binom{n}{2}x^{n-2}h^2 + \cdots + \binom{n}{n-2}x^2h^{n-2} + \binom{n}{n-1}xh^{n-1} + h^n}{h}$$

$$= nx^{n-1} + \binom{n}{2}x^{n-2}h + \cdots + \binom{n}{n-2}x^2h^{n-3} + \binom{n}{n-1}xh^{n-2} + h^{n-1}$$

So $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = nx^{n-1}$

$f'(2) = n \cdot 2^{n-1}$

Remark Sometimes people also write $\frac{df(x)}{dx}$ or $\frac{d}{dx}f(x)$ to mean $f'(x)$

Example Find the slope of the tangent to the graph of $f$ at $x=3$.
where $f(x) = x^3 + x^2$

The slope of the tangent to the graph of $f$ at $x=3$ is $f'(3)$.
so we only need to find $f'(3)$

$$f(x+h) = (x+h)^3 + (x+h)^2 = x^3 + 3x^2h + 3xh^2 + h^3 + x^2 + 2xh + h^2$$

$$\frac{f(x+h)-f(x)}{h} = \frac{3x^2h + 3xh^2 + h^3 + 2xh + h^2}{h} = 3x^2 + 3xh + h^2 + 2x + h$$

So $f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}(3x^2 + 3xh + h^2 + 2x + h) = 3x^2 + 2x$

$f'(3) = 3\times 3^2 + 2\times 3 = 33$