Monotonic Functions

f is a function. f is increasing if $x_1 < x_2$ implies $f(x_1) \le f(x_2)$.

f is strictly increasing if $x_1 < x_2$ implies $f(x_1) < f(x_2)$.

f is decreasing if $x_1 < x_2$ implies $f(x_1) \ge f(x_2)$

f is strictly decreasing if $x_1 < x_2$ implies $f(x_1) > f(x_2)$.

Example. $f(x) = x^3$ is a strictly increasing function.

proof. for any $x_1 < x_2$.

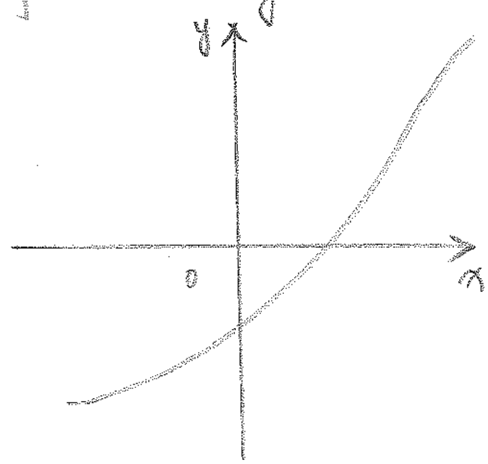
$$f(x_1) - f(x_2) = x_1^3 - x_2^3 = (x_1 - x_2)(x_1^2 + x_2^2 + x_1 x_2)$$

$$= (x_1 - x_2)\left[\left(x_1 + \frac{x_2}{2}\right)^2 + \frac{3}{4}x_2^2\right]$$

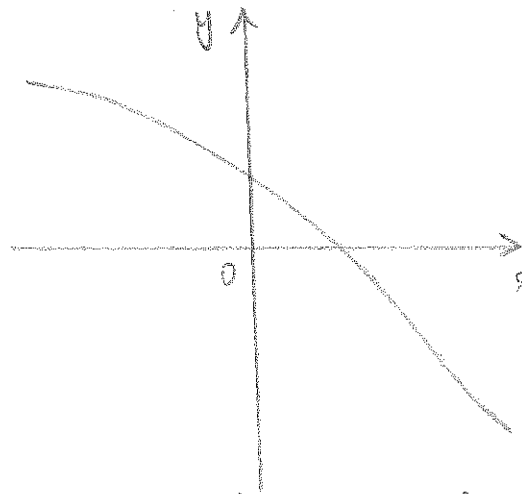
$$< 0$$

i.e. $f(x_1) < f(x_2)$.

Graph of increasing & decreasing functions:



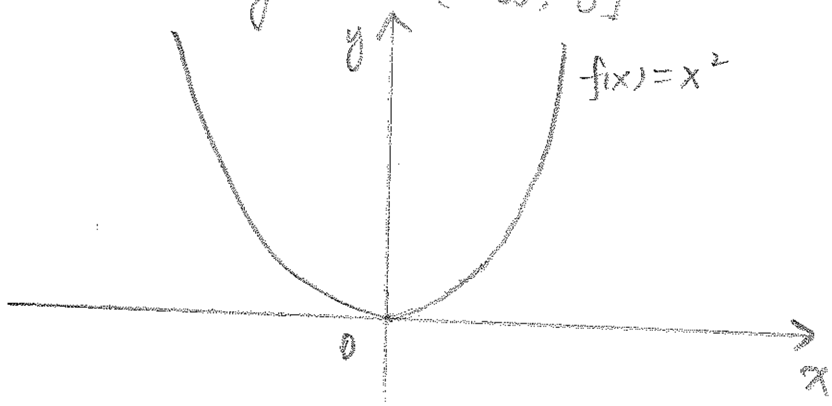
strictly increasing function
going "SW-NE" direction



strictly decreasing function
going "NW-SE" direction.

Remark. A function may not be increasing or decreasing on the whole domain, but when restricted to smaller intervals, it may be increasing or decreasing.

Example. The function $f(x) = x^2$ is increasing on $[0, +\infty)$ and decreasing on $(-\infty, 0]$



Question. Is the function $f(x) \equiv 0$ increasing? Strictly increasing? decreasing? Strictly decreasing?

Theorem. A strictly increasing function / strictly decreasing function is a one-to-one function.

Corollary. A strictly increasing function / strictly decreasing function has an inverse function

Limit & Rate of Change

f is a function. If $f(x)$ tends to A as x tends to a. we say A is the limit of f as x tends to a. written as $\lim\limits_{x\to a} f(x) = A$

Example $f(x) = \frac{1}{x}$. as x tends to $+\infty$. $f(x)$ tends to 0.

so $\lim\limits_{x\to +\infty} f(x) = 0$

Example. $f(x) = x^2$. as x tends to 3. $f(x)$ tends to 9

so $\lim\limits_{x\to 3} f(x) = 9$

Remark. • $\lim\limits_{x\to a} f(x)$ depends on the behavior of f near a. but not on the value at $x = a$

• If a is a finite number. "x tends to a" means x tends to a from both directions.

Theorem. If $\lim\limits_{x\to a} f(x) = A$. and $\lim\limits_{x\to a} g(x) = B$ then

(1). $\lim\limits_{x\to a} (f(x) \pm g(x)) = A \pm B$

(2) $\lim\limits_{x\to a} (f(x) \cdot g(x)) = AB$

(3) $\lim\limits_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{A}{B}$ (if $B \neq 0$)

(4) $\lim\limits_{x\to a} (f(x))^r = A^r$ (if A^r is defined and r is a real number)

Example Find $\lim_{x\to 4} \frac{2x^{\frac{3}{2}} - \sqrt{x}}{x^2-15}$

$$\lim_{x\to 4} \frac{2x^{\frac{3}{2}} - \sqrt{x}}{x^2-15} = \frac{\lim_{x\to 4}(2x^{\frac{3}{2}} - x^{\frac{1}{2}})}{\lim_{x\to 4}(x^2-15)} = \frac{2(\lim_{x\to 4} x)^{\frac{3}{2}} - (\lim_{x\to 4} x)^{\frac{1}{2}}}{(\lim_{x\to 4} x)^2 - 15}$$

$$= \frac{2 \times 4^{\frac{3}{2}} - 4^{\frac{1}{2}}}{4^2 - 15} = 14$$

Example. Find $\lim_{x\to 3} \frac{x^2-9}{x-3}$

We cannot write $\lim_{x\to 3} \frac{x^2-9}{x-3} = \frac{\lim_{x\to 3}(x^2-9)}{\lim_{x\to 3}(x-3)}$

since $\lim_{x\to 3}(x-3) = 0$

Instead, we will do the following:

$$\lim_{x\to 3} \frac{x^2-9}{x-3} = \lim_{x\to 3} \frac{(x-3)(x+3)}{x-3} = \lim_{x\to 3}(x+3) = 6$$

Example. Find $\lim_{x\to 0} \frac{\sqrt{x+1}-1}{x}$

$$\lim_{x\to 0} \frac{\sqrt{x+1}-1}{x} = \lim_{x\to 0} \frac{(\sqrt{x+1}-1)(\sqrt{x+1}+1)}{x(\sqrt{x+1}+1)} = \lim_{x\to 0} \frac{x}{x(\sqrt{x+1}+1)} = \lim_{x\to 0} \frac{1}{\sqrt{x+1}+1}$$

$$= \frac{1}{2}$$

Note: We can multiply $\sqrt{x+1}+1$ on both the denominator and the numerator because $\sqrt{x+1}+1$ does not equal to 0 when x is close to 0

f is a function. The rate of change of f at $x=a$ is defined to be the number $\lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$. We also call it the instantaneous rate of change of f at a, denoted as:

$$f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$$

We define the relative rate of change of f at a to be $\frac{f'(a)}{f(a)}$

It's also called proportional rate of change.

We call $\frac{f(a+h)-f(a)}{h}$ the average rate of change of f over the interval $[a, a+h]$ (if $h>0$) or $[a+h, a]$ (if $h<0$)

Application in Economics:

$C(x)$ is the cost of producing x units of goods. We call the instantaneous rate of change $C'(a)$ the marginal cost at a.

Intuitively, when h is a small number, $C'(a) \approx \frac{C(a+h)-C(a)}{h}$

$\Rightarrow C(a+h)-C(a) \approx C'(a)h$

So the incremental cost of producing h units of extra output can be approximated by the number $C'(a)h$ when h is small.

Intuitively, for example, if $a=5{,}000$, $C(a) = \$700{,}000$, $C'(a) = 170$, it means if we produce 5,000 goods, the cost is \$700,000, and the cost will increase by about \$170 if we produce 1 more goods besides the 5,000.