FUNCTIONS

A function of a real variable x with domain D is a rule that assigns a unique real number to each real number x in D.

As x varies over the whole domain, the set of all possible resulting values $f(x)$ is called the range of f

The value assigned to x by the rule f is denoted as $f(x)$

Examples ① $f(x) = x$ defined on $\mathbb{R}$. the Identity Function

② $f(x) = |x|$ defined on $\mathbb{R}$. the Absolute Value Function

③ $f(x) = x^2 + 2x + 3$ defined on $\mathbb{R}$.

④ $f(x) = \frac{1}{x-1}$ defined on $(-\infty, 1) \cup (1, +\infty)$

⑤ If $f(x) = x^2+1$, then $f(3) = 3^2 + 1 = 10$

Translate descriptive definition of a function into mathematical expression:

Example. The volume of a cube is the third power of its edge length.

So if the length of the edge of a cube is x, the volume is a function of x given by

$$V(x) = x^3$$

Domain of a function:

The domain of a function $f(x)$ is the set of numbers from which x can take value of.

If the domain is not specified for a given function, we regard the domain to be the set of all values for which the function gives a unique value.

Example. Find the domain of

① $f(x) = \frac{1}{x-1}$

$x-1 \neq 0 \Rightarrow x \neq 1$ so the domain is $(-\infty, 1) \cup (1, +\infty)$

② $f(x) = \sqrt{x-1}$

$x-1 \geq 0 \Rightarrow x \geq 1$ so the domain is $[1, +\infty)$

Range of a function:

The set of all values $f(x)$ that the function assumes is called the range of f.

Example. Find the range of $f(x) = |x|$.

Since the absolute value of a real number is nonnegative, the range of f is $[0, +\infty)$

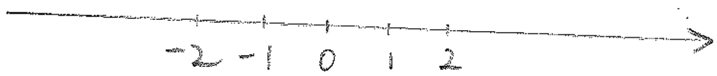
Example. Find the range of $f(x) = |x| + 1$

By previous example, the range of $f(x) = |x|$ is $[0, +\infty)$

so the range of $f(x) = |x| + 1$ is $[1, +\infty)$

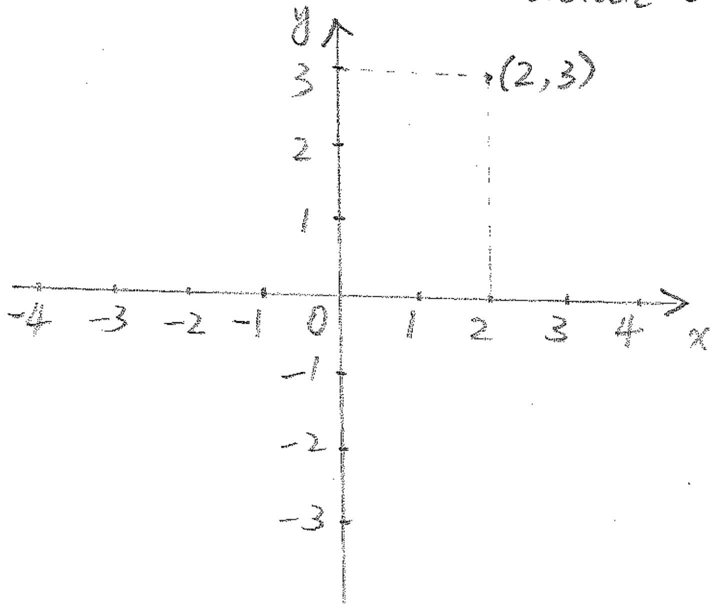
Graphs of Functions

Recall that a real line is an oriented straight line, such that points on the real line are in one-to-one correspondence with the real numbers. The number corresponding to zero is called the origin, denoted as O



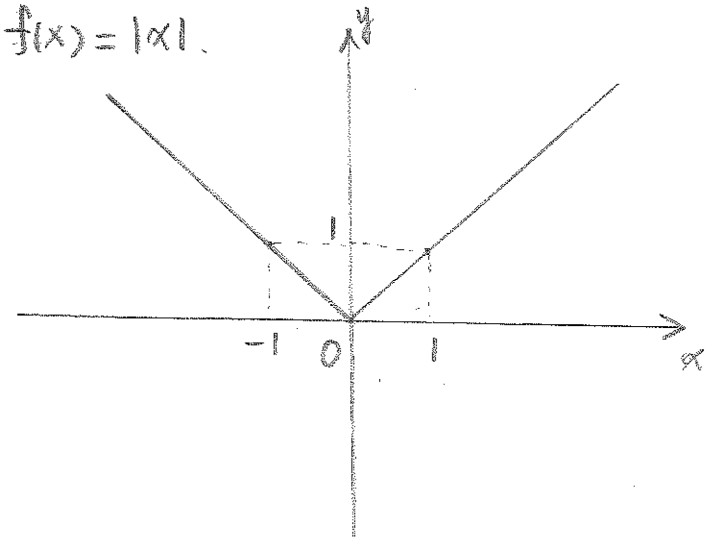
Real Line

Now we take two copies of the real line, putting them on the plane in the way that one of them horizontally pointing to the right, the other vertically pointing upward, and their origin coincide
The horizontal line is called the x-axis, and the vertical line is called the y-axis. Then every point on the plane can be uniquely represented as a pair of real numbers (x, y) by projecting to x-axis and y-axis.
The above construction is called a Cartisian Coordinate System.

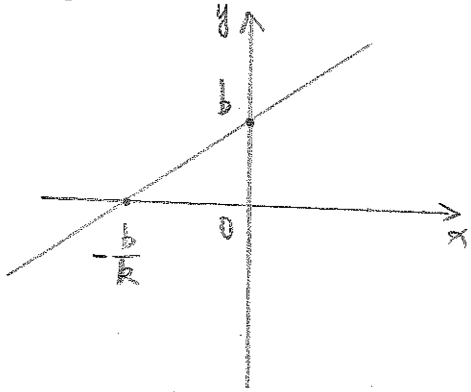


The graph of a function f is the set of all the points $(x, f(x))$ in the coordinate system.

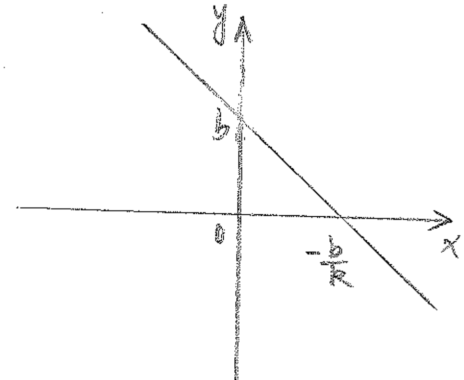
Example. $f(x) = |x|$.



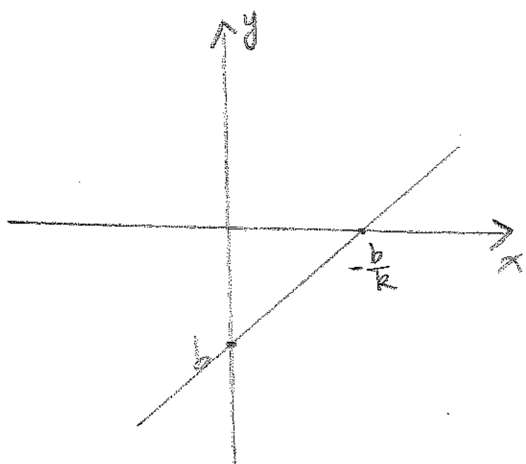
Example. $f(x) = kx + b$. (k, b are constants)



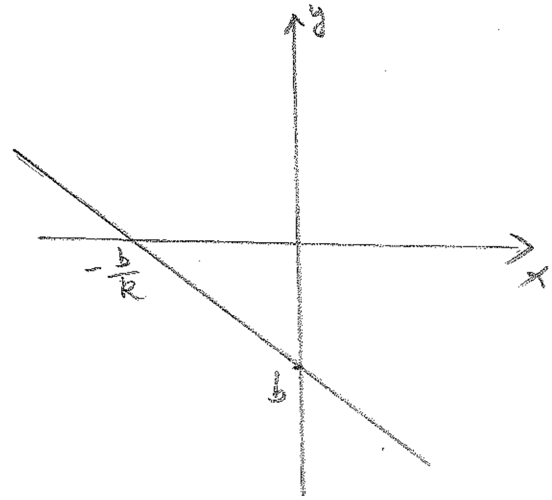
$k > 0$, $b > 0$



$k < 0$, $b > 0$



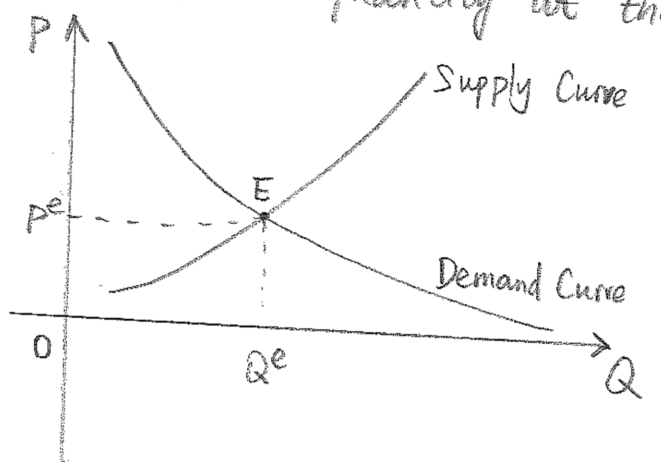
$k > 0$, $b < 0$



$k < 0$, $b < 0$

Example. Supply and Demand

For some kinds of goods, as the price P is getting higher, the demand will decrease and the supply will increase.
We regard both demand and supply as functions of price P, then we can plot their graphs on the coordinate system.
The graphs are called demand curve and supply curve.
For historical reasons, people usually draw the price at the vertical line and the quantity at the horizontal line.



A point where the two curves intersect is called an equilibrium.
This gives a price P^e at which consumers will buy the same amount of the goods as the producers will to sell.

Inverse Function

f is a function with domain A and range B.

We say f is one-to-one in A if f never assign a same value to two different numbers in A. In other words, for any $x_1 \neq x_2$ in the domain, $f(x_1) \neq f(x_2)$.

When f is one-to-one, it has an inverse function g with domain B and range A. satisfying: For each y in B, the value $g(y)$ is the unique number x in A such that $f(x)=y$.

i.e. $g(y)=x \Leftrightarrow y=f(x)$ $(x \in A,\ y \in B)$

How to find the inverse function of a given function f?

Let $y=f(x)$. Then solve for x regarding y as the independent variable. Figure out the domain of the inverse function by computing the range of f.

Example. Find the inverse function for $f(x)=2x+3$.

Denote g to be the inverse function of f. Since the range of f is $\mathbb{R}$, the domain of g is $\mathbb{R}$.

$$y=f(x)=2x+3 \Rightarrow x=\frac{y-3}{2}$$

so $g(y)=\dfrac{y-3}{2}$

Example. Find the inverse function of $f(x)=\sqrt{x-3}$.

Denote g to be the inverse function of f.

The range of f is $[0,+\infty)$ so the domain of g is $[0,+\infty)$

$y=f(x)=\sqrt{x-3} \Rightarrow x=y^2+3$.

So $g(y)=y^2+3$

Graph of Inverse Function.

Observe that if (x,y) is on the graph of a one-to-one function f.
then (y,x) must be on the graph of the inverse function of f.
And (x,y), (y,x) are symmetric with respect to the line $y=x$.
So the graph of a one-to-one function and its inverse function are symmetric with respect to the line $y=x$