Optimization for Functions of Two-Variables

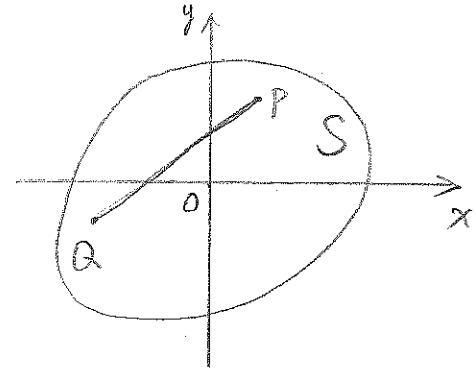
$z=f(x,y)$ is a function of two-variables, defined on a set S.
We are going to study abod the extreme points and extreme values of the function, which are defined in a same manner as those for single-variable functions.

If f obtains maximum or minimum at an interior point of S, say (x_0, y_0), then consider the functions $f(x, y_0)$ and $f(x_0, y)$, for the former one, $x = x_0$ is an extreme point and for the latter one, $y = y_0$ is an extreme point, so $\frac{\partial f}{\partial x}(x_0, y_0) = \frac{\partial f}{\partial y}(x_0, y_0) = 0$.
We have obtained a necessary condition for an interior point to be an extreme point:
Theorem. If $z=f(x,y)$, a differentiable function, takes extreme at (x_0, y_0) which is an interior point, then $\frac{\partial f}{\partial x}(x_0, y_0) = \frac{\partial f}{\partial y}(x_0, y_0) = 0$.

Next we are looking for sufficient conditions for being an extreme point.

We call a set S in the x-y plane is convex if for each pair of points P, Q in S, all the line segment between P and Q lies in S.

In particular, we consider the whole plane to be a convex set.



If we consider a function defined on a convex set, there is a method to tell if a point is an extreme:

Theorem. Suppose (x_0, y_0) is an interior point for a twice-differentiable function $f(x,y)$ defined on a convex set S in $\mathbb{R}^2$ such that

$\frac{\partial f}{\partial x}(x_0, y_0) = 0$ and $\frac{\partial f}{\partial y}(x_0, y_0) = 0$ Then:

(a). If for all $(x,y) \in S$,

$$\frac{\partial^2 f}{\partial x^2}(x,y) \leq 0, \quad \frac{\partial^2 f}{\partial y^2}(x,y) \leq 0, \quad \frac{\partial^2 f}{\partial x^2}(x,y)\frac{\partial^2 f}{\partial y^2}(x,y) - \left(\frac{\partial^2 f}{\partial x \partial y}(x,y)\right)^2 \geq 0.$$

then (x_0, y_0) is a maximum point for $f(x,y)$ in S.

(b). If for all $(x,y) \in S$

$$\frac{\partial^2 f}{\partial x^2}(x,y) \geq 0, \quad \frac{\partial^2 f}{\partial y^2}(x,y) \geq 0, \quad \frac{\partial^2 f}{\partial x^2}(x,y)\frac{\partial^2 f}{\partial y^2}(x,y) - \left(\frac{\partial^2 f}{\partial x \partial y}(x,y)\right)^2 \geq 0.$$

then (x_0, y_0) is a minimum point for $f(x,y)$ in S.

A remark is that if you know some linear algebra, you can view:

$$\frac{\partial^2 f}{\partial x^2}(x,y)\cdot\frac{\partial^2 f}{\partial y^2}(x,y) - \left(\frac{\partial^2 f}{\partial x \partial y}(x,y)\,\frac{\partial^2 f}{\partial y \partial x}(x,y)\right) = \det\begin{bmatrix} \frac{\partial^2 f}{\partial x^2}(x,y) & \frac{\partial^2 f}{\partial x \partial y}(x,y) \\ \frac{\partial^2 f}{\partial y \partial x}(x,y) & \frac{\partial^2 f}{\partial y^2}(x,y) \end{bmatrix}$$

The matrix is called ther Hersian matrix for the function f.

Example. Find the maximum points of $f(x,y) = -2x^2 - 2xy - 2y^2 + 36x + 42y - 158$

$$\begin{cases} \frac{\partial f}{\partial x} = -4x - 2y + 36 \\ \frac{\partial f}{\partial y} = -4y - 2x + 42 \end{cases}$$

let $\frac{\partial f}{\partial x} = 0$, $\frac{\partial f}{\partial y} = 0$. we get

$$\begin{cases} -4x - 2y + 36 = 0 \\ -4y - 2x + 42 = 0 \end{cases} \Rightarrow \begin{cases} x = 5 \\ y = 8 \end{cases}$$

For all $(x,y) \in \mathbb{R}^2$:

$$\frac{\partial^2 f}{\partial x^2} = -4 < 0, \quad \frac{\partial^2 f}{\partial y^2} = -4 < 0. \quad \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}(-4y - 2x + 42) = -2.$$

so $\frac{\partial^2 f}{\partial x^2}\cdot\frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 = 12 > 0$

We conclude $(5, 8)$ is a maximum point.

Example. A firm has 3 factories each producing the same item.

Let x, y, z denote the respective output that the three factories produce in order to fulfill an order for 2000 units in total, i.e. $x+y+z=2000$. The cost for the 3 factories are:

$$C_1(x) = 200 + \frac{1}{100}x^2, \quad C_2(y) = 200 + y + \frac{1}{300}y^3, \quad C_3(z) = 200 + 10z$$

Find (x, y, z) that minimizes total cost.

$$C(x, y, z) = C_1(x) + C_2(y) + C_3(z).$$

Since $z = 2000 - x - y$, the total cost is a function of (x, y):

$$\begin{aligned}
\tilde{C}(x,y) &= C(x, y, 2000-x-y) \\
&= C_1(x) + C_2(y) + C_3(2000-x-y) \\
&= 200 + \frac{1}{100}x^2 + 200 + y + \frac{1}{300}y^3 + 200 + 10(2000-x-y) \\
&= \frac{1}{100}x^2 - 10x + \frac{1}{300}y^3 - 9y + 20600
\end{aligned}$$

$$\begin{cases} \dfrac{\partial \tilde{C}(x,y)}{\partial x} = \dfrac{1}{50}x - 10 \\[2ex] \dfrac{\partial \tilde{C}(x,y)}{\partial y} = \dfrac{1}{100}y^2 - 9 \end{cases}$$

let $\dfrac{\partial \tilde{C}(x,y)}{\partial x} = \dfrac{\partial \tilde{C}(x,y)}{\partial y} = 0$

we get $x = 500$, $y = 30$

$$\frac{\partial^2 \tilde{C}}{\partial x^2} = \frac{1}{50} > 0, \quad \frac{\partial^2 \tilde{C}}{\partial y^2} = \frac{1}{50}y \geq 0 \ (\text{since } y \geq 0).$$

$$\frac{\partial^2 \tilde{C}}{\partial x^2} \cdot \frac{\partial^2 \tilde{C}}{\partial y^2} - \left(\frac{\partial^2 \tilde{C}}{\partial x \partial y}\right)^2 = \frac{1}{50} \cdot \frac{1}{50}y - 0 = \frac{1}{2500}y \geq 0$$

so $\tilde{C}(x,y)$ obtains minimum at $x = 500$, $y = 30$.

The cost is minimized when $x = 500$, $y = 30$, $z = 1470$

For a function $f(x,y)$, we say (x_0, y_0) is a local maximum if $f(x_0, y_0) \geq f(x, y)$ for all (x, y) that are sufficiently close to (x_0, y_0). (x_0, y_0) is a local minimum if $f(x_0, y_0) \leq f(x, y)$ for all (x, y) that are sufficiently close to (x_0, y_0)

Global extremes are local extremes, while local extremes may not be global extremes.

Local extremes satisfy a same necessary condition as global extremes:

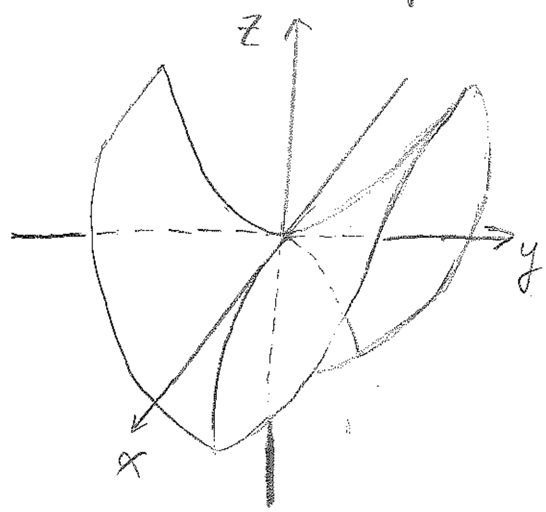
Theorem. If (x_0, y_0) is an interior point in S, which is a local extreme of $f(x, y)$, then $\frac{\partial f}{\partial x}(x_0, y_0) = 0$, $\frac{\partial f}{\partial y}(x_0, y_0) = 0$

Again, we need more conditions to conclude a point is a local extreme.

If (x_0, y_0) satisfies $\frac{\partial f}{\partial x}(x_0, y_0) = 0$ and $\frac{\partial f}{\partial y}(x_0, y_0) = 0$, but (x_0, y_0) is not a local extreme, we say (x_0, y_0) is a saddle point of f.

So if $\frac{\partial f}{\partial x}(x_0, y_0) = 0$ and $\frac{\partial f}{\partial y}(x_0, y_0) = 0$, there are 3 possibilities:

- (x_0, y_0) is a local maximum
- (x_0, y_0) is a local minimum
- (x_0, y_0) is a saddle point



Theorem. $f(x, y)$ is a twice-differentiable function in a set S, and let (x_0, y_0) be an interior point of S satisfying

$\frac{\partial f}{\partial x}(x_0, y_0) = 0$, $\frac{\partial f}{\partial y}(x_0, y_0) = 0$.

(a). If $\frac{\partial^2 f}{\partial x^2}(x_0, y_0) < 0$ and $\frac{\partial^2 f}{\partial x^2} \cdot \frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 > 0$, then (x_0, y_0) is a local maximum.

(b). If $\frac{\partial^2 f}{\partial x^2}(x_0, y_0) > 0$ and $\frac{\partial^2 f}{\partial x^2} \cdot \frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 > 0$, then (x_0, y_0) is a local minimum.

(c) If $\frac{\partial^2 f}{\partial x^2} \cdot \frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 < 0$, then (x_0, y_0) is a saddle point.

Example. $f(x, y) = x^3 - x^2 - y^2 + 8$

$\begin{cases} \frac{\partial f}{\partial x} = 3x^2 - 2x \\ \frac{\partial f}{\partial y} = -2y \end{cases}$ let $\begin{cases} \frac{\partial f}{\partial x} = 0 \\ \frac{\partial f}{\partial y} = 0 \end{cases}$ we get $x = 0$ or $\frac{2}{3}$, $y = 0$

so the critical points are $(0, 0)$ and $(\frac{2}{3}, 0)$

$\frac{\partial^2 f}{\partial x^2} = 6x - 2$, $\frac{\partial^2 f}{\partial y^2} = -2$, $\frac{\partial^2 f}{\partial x \partial y} = 0$. so

$\frac{\partial^2 f}{\partial x^2}(0,0) = -2$, $\frac{\partial^2 f}{\partial x^2}(0,0) \cdot \frac{\partial^2 f}{\partial y^2}(0,0) - \left(\frac{\partial^2 f}{\partial x \partial y}(0,0)\right)^2 = (-2) \times (-2) - 0 = 4$

$(0, 0)$ is a local maximum point.

$\frac{\partial^2 f}{\partial x^2}(\frac{2}{3}, 0) = 2$, $\frac{\partial^2 f}{\partial x^2}(\frac{2}{3}, 0) \cdot \frac{\partial^2 f}{\partial y^2}(\frac{2}{3}, 0) - \left(\frac{\partial^2 f}{\partial x \partial y}(0,0)\right)^2 = 2 \times (-2) - 0 = -4$

$(\frac{2}{3}, 0)$ is a saddle point.