Linear Approximations & Differential

If $f(x,y)$ is a differentiable function, then at (x_0, y_0), the equation of the tangent plane is given by $z - f(x_0,y_0) = \frac{\partial f}{\partial x}(x_0,y_0)(x-x_0) + \frac{\partial f}{\partial y}(x_0,y_0)(y-y_0)$

The linear approximation of a function $f(x,y)$ at (x_0, y_0) is to use the equation of the tangent plane of the graph of the function at (x_0, y_0) to approximate the function, i.e. the linear approximation to $f(x,y)$ around (x_0, y_0) is: $f(x,y) \approx f(x_0,y_0) + \frac{\partial f}{\partial x}(x_0,y_0)(x-x_0) + \frac{\partial f}{\partial y}(x_0,y_0)(y-y_0)$

Example. Find the linear approximation to $f(x,y) = e^{x+y}(xy-1)$ about $(0,0)$

$f(0,0) = -1$. $\quad \frac{\partial f}{\partial x} = e^{x+y}(xy-1) + e^{x+y}y = e^{x+y}(xy-1+y)$

$\frac{\partial f}{\partial y} = e^{x+y}(xy-1) + e^{x+y}x = e^{x+y}(xy-1+x)$

so $\frac{\partial f}{\partial x}(0,0) = -1$, $\frac{\partial f}{\partial y}(0,0) = -1$

so $f(x,y) \approx -1 - x - y$ near $(0,0)$

Example. $f(x,y) = xy^3 - 2x^3$. approximate $f(2.01, 2.98)$ by linear approximation

$f(2,3) = 38$, $\frac{\partial f}{\partial x} = y^3 - 6x^2$, $\frac{\partial f}{\partial y} = 3xy^2$

so $\frac{\partial f}{\partial x}(2,3) = 3^3 - 6\times 2^2 = 3$, $\frac{\partial f}{\partial y}(2,3) = 3\times 2\times 3^2 = 54$. so

$f(x,y) \approx 38 + 3(x-2) + 54(y-3)$ near $(2,3)$

so $f(2.01, 2.98) \approx 38 + 3(2.01-2) + 54(2.98-3)$

$= 38 + 3\times 0.01 - 54\times 0.02$

$= 36.95$

Example. Find the tangent plane at $P=(1,1,5)$ to the graph of $f(x,y)=x^2+2xy+2y^2$

$\frac{\partial f}{\partial x}=2x+2y$, $\frac{\partial f}{\partial y}=2x+4y$, so $\frac{\partial f}{\partial x}(1,1)=4$, $\frac{\partial f}{\partial y}(1,1)=6$

So the tangent plane is $z-5=4(x-1)+6(y-1)$

For a function $f(x,y)$, we define the differential of $z=f(x,y)$ at (x_0,y_0) to be $dz=df=\frac{\partial f}{\partial x}(x_0,y_0)\,dx+\frac{\partial f}{\partial y}(x_0,y_0)\,dy$.

Linear approximation tells us that when (x,y) is close to (x_0,y_0),

$$f(x,y)-f(x_0,y_0)\approx df=\frac{\partial f}{\partial x}(x_0,y_0)\,dx+\frac{\partial f}{\partial y}(x_0,y_0)\,dy$$

where $dx=x-x_0$, $dy=y-y_0$

Theorem. $f(x,y)$ and $g(x,y)$ are differentiable functions, then:

① $d(af+bg)=a\,df+b\,dg$

② $d(fg)=g\,df+f\,dg$

③ $d\left(\frac{f}{g}\right)=\frac{g\,df-f\,dg}{g^2}$ $(g\neq 0)$

Example. Find an expression for dz in terms of dx and dy for $z=e^{xu}$ where $u=u(x,y)$.

$$dz=de^{xu}=\frac{\partial e^{xu}}{\partial x}dx+\frac{\partial e^{xu}}{\partial y}dy=\left(u+x\cdot\frac{\partial u}{\partial x}\right)e^{xu}dx+x\frac{\partial u}{\partial y}e^{xu}dy$$

Theorem. If $z=g(f(x,y))$, then $dz=g'(f(x,y))\,df$.

Example. Find an expression for dz in terms of dx and dy for $z=e^{xu}$ where $u=u(x,y)$

$$dz=de^{xu}=e^{xu}d\,xu=e^{xu}\left(\left(u+x\frac{\partial u}{\partial x}\right)dx+x\frac{\partial u}{\partial y}dy\right)$$