Implicit Differentiation Along A Level Curve

We have discussed about how to take the derivative of a function which is implicitely defined by some equation. Recall that the idea is to differentiate both sides of the equation with respect to x, keeping in mind that y is a function of x.

Now we have a new way to look at the question of implicit differentiation:

Let $F(x,y)$ be a function of two variables, and consider the level curve $F(x,y)=C$ (C is a constant). Suppose y is a function of x on part of the level curve, say $y=f(x)$. Then we can find $y'=f'(x)$ by applying what we have learned about multi-variable calculus: $F(x, f(x))=C$

Take derivative with respect to x on both sides, by the chain rule, we get

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\cdot\frac{dy}{dx} = 0$$

so when $\frac{\partial F}{\partial y} \neq 0$, we have : $\frac{dy}{dx} = -\dfrac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}}$

Example. Find the slope and equation for the tangent line at $(x,y)=(2,1)$ to the curve $x^3+x^2y-2y^2-10y=0$

Let $F(x,y)=x^3+x^2y-2y^2-10y$ then we are considering the level set $F(x,y)=0$.

$$\frac{\partial F}{\partial x} = 3x^2 + 2xy, \quad \frac{\partial F}{\partial y} = x^2 - 4y - 10$$

So $\frac{\partial F}{\partial x}(2,1) = 3\times 2^2 + 2\times 2\times 1 = 16$

$\frac{\partial F}{\partial y}(2,1) = 2^2 - 4\times 1 - 10 = -10$

$$y' = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}} = -\frac{16}{-10} = \frac{8}{5}$$

So the slope is $\frac{8}{5}$, the tangent line is $y-1 = \frac{8}{5}(x-2)$

Example. The demand D for a commodity is a function of the price P before tax and the sales tax per unit, t. so $D = f(t,p)$.

Suppose S is the supply, which is a function of price P.

i.e. $S = g(P)$.

At equilibrium, when supply equals demand. the equilibrium price $P = P(t)$ is a function of the sales tax t.

We would like to investigate the influence of sales tax on the equilibrium price, and their relation is implicitly given by the equilibrium condition $D = S$

i.e. $f(t,p) = g(p)$

Let $F(t,P) = f(t,p) - g(p)$. the equilibrium corresponds to

$F(t,p) = f(t,p) - g(p) = 0$.

$$\frac{\partial F}{\partial t} = \frac{\partial f}{\partial t}, \quad \frac{\partial F}{\partial p} = \frac{\partial f}{\partial p} - \frac{dg}{dp}$$

Now we consider P as a function of t on the level set $F(t,p)=0$, and taking implicite differentiation:

$$\frac{dP}{dt} = -\frac{\frac{\partial F}{\partial t}}{\frac{\partial F}{\partial p}} = -\frac{\frac{\partial f}{\partial t}}{\frac{\partial f}{\partial p} - \frac{dg}{dp}} = \frac{\frac{\partial f}{\partial t}}{\frac{dg}{dp} - \frac{\partial f}{\partial p}}$$

Since $S=g(p)$ is the supply function, we usually assume it's strictly increasing, i.e. $\frac{dg}{dp} > 0$. (increase in price encourages merchants)

$D=f(t,p)$ is the demand function, we usually assume $\frac{\partial f}{\partial t}<0$, $\frac{\partial f}{\partial p}<0$ (increase in price or tax discourages consumers)

So putting all these assumptions together, we see $\frac{dP}{dt} < 0$ which means an increase in sales tax will make the equilibrium price to drop.

Implicit Differentiation with One More Variable:

Consider a function $F(x,y,z)$ of three variables. For a constant c, the set of points in the space satisfying $F(x,y,z)=C$ forms a surface. Suppose that z is a function of x and y on this surface, $z=f(x,y)$. Then we can use implicit differentiation to find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$:

$$F(x,y,f(x,y)) = C, \quad \text{when } \frac{\partial F}{\partial z} \neq 0$$

So by Chain Rule,

$$\begin{cases} \frac{\partial F}{\partial x} + \frac{\partial F}{\partial z}\cdot\frac{\partial z}{\partial x} = 0 \\ \frac{\partial F}{\partial y} + \frac{\partial F}{\partial z}\cdot\frac{\partial z}{\partial y} = 0 \end{cases} \Rightarrow \begin{cases} \frac{\partial z}{\partial x} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}} \\ \frac{\partial z}{\partial y} = -\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}} \end{cases}$$

Example. Find the partial derivatives $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if z is implicitly defined as a function of x and y via:

$$x^2+y^2+z^2=1$$

Let $F(x,y,z)=x^2+y^2+z^2$, then $\frac{\partial F}{\partial x}=2x$, $\frac{\partial F}{\partial y}=2y$. $\frac{\partial F}{\partial z}=2z$

z is defined as a function of x & y by $F(x,y,z)=1$, so

$$\frac{\partial z}{\partial x}=-\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}}=-\frac{2x}{2z}=-\frac{x}{z}$$

$$\frac{\partial z}{\partial y}=-\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}}=-\frac{2y}{2z}=-\frac{y}{z}$$

Example. A firm produces $Q=f(L)$ units of a commodity, using L units of Labor. Assume $f'(L)>0$ and $f''(L)\le 0$. If the firm gets P dollars per unit produced and pays w dollars for a unit of labor, write down the profit function, and find the first-order condition for profit maximization at $L^*>0$. Then by implicit differentiation of the first-order condition, examine how changes in P and w affect the optimal choice L^*.

The profit function $\pi(L)=\underbrace{Pf(L)}_{\text{revenue}}-\underbrace{wL}_{\text{cost}}$

$\pi'(L)=Pf'(L)-w$. When profit is maximized at L^*, we have

$\pi'(L^*)=Pf'(L^*)-w=0$

Let $F(P,w,L^*)=Pf'(L^*)-w$, then implicit differentiation gives

$$\frac{\partial L^*}{\partial P}=-\frac{\frac{\partial F}{\partial P}}{\frac{\partial F}{\partial L^*}}=-\frac{f'(L^*)}{Pf''(L^*)} \qquad \frac{\partial L^*}{\partial w}=-\frac{\frac{\partial F}{\partial w}}{\frac{\partial F}{\partial L^*}}=-\frac{-1}{Pf''(L^*)}=\frac{1}{Pf''(L^*)}$$

We see that since $f'(L)>0$, $f''(L)<0$.

$$\frac{\partial L^*}{\partial p} > 0 \quad \text{and} \quad \frac{\partial L^*}{\partial w} < 0.$$

So if price increases, the optimal labor input goes up.

if labor cost increases, the optimal labor input goes down.

Marginal Rate of Substitution (MRS):

If $F(x,y)$ is a function of two variables, we define

$$R_{yx} = \frac{\frac{\partial F}{\partial x}(x,y)}{\frac{\partial F}{\partial y}(x,y)}$$ to be the marginal rate of substitution.

If we consider a level set $F(x,y)=c$, and assume y is a function of x on this level set, we get:

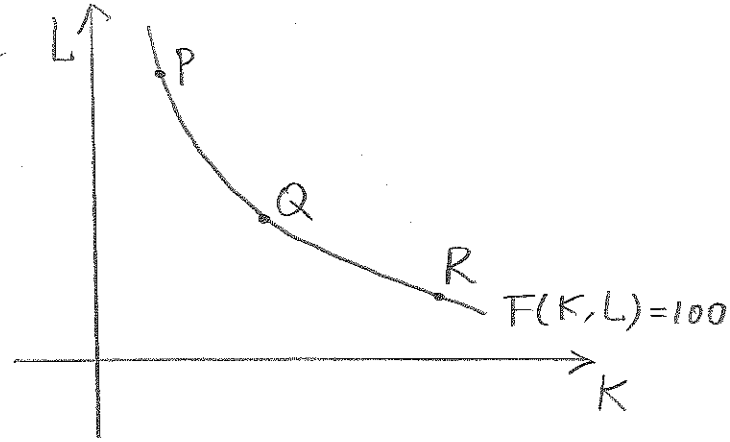
$$R_{yx} = \frac{\frac{\partial F}{\partial x}(x,y)}{\frac{\partial F}{\partial y}(x,y)} = -\left(-\frac{\frac{\partial F}{\partial x}(x,y)}{\frac{\partial F}{\partial y}(x,y)}\right) = -\frac{dy}{dx}$$

This implies $dy = -R_{yx}\,dx$

Recall that dy is the approximation of the change of y when x is changed by dx, so R_{yx} is approximately the quantity of y we must substitute (add) per unit of x removed, if we are to stay on the same level curve.

Example. Let $F(K,L)=100$ be an isoquant for a production function $F(K,L)$, where K is capital input and L is labor input, 100 is the output. Look at the following graph:

This is the level set corresponding to $F(K,L)=100$.



At point P, the curve is very steep, so it indicates if capital is reduced by a small amount, then we need much more increase in labor to keep output at 100.

Say if the slope of the curve is -4, i.e $\frac{dL}{dK}=-4$ at P, then $R_{LK}(P) = -\frac{dL}{dK} = 4$, so if we remove capital by 1 unit, we need to add the labor by about 4 units to keep the output.

Q is the point at which the slope is -1. so $R_{LK}(Q)=1$, if we remove capital by 1 unit, we need to add the labor by about 1 unit to keep the output.

At point R, the curve is nearly flat, so it indicates we only need to increase labor by relatively small amount in order to keep the output when the capital is reduced.

Say if the slope of the curve is $-\frac{1}{4}$, i.e. $\frac{dL}{dK} = -\frac{1}{4}$ at R, then $R_{LK}(R) = \frac{1}{4}$, so if we remove capital by 1 unit, we need to add the labor by about $\frac{1}{4}$ units to keep the output

When the level set is a convex curve (like that in the previous example), we see as we move along the level curve from left to right, the MRS R_{yx} is strictly decreasing, so for each value of R_{yx}, there is a corresponding point (x,y) on the level set.

then we have a corresponding value $\frac{y}{x}$. So this fraction $\frac{y}{x}$ is a function of R_{yx}, which motivates the following:

When $F(x,y)=C$, the elasticity of substitution bewteen y and x is defined by $\sigma_{yx} = El_{R_{yx}}\left(\frac{y}{x}\right)$

The intuition of this definition σ_{yx} is that σ_{yx} is roughly the percentage change in $\frac{y}{x}$ if R_{yx} is increased by 1%

Example. The Cobb-Douglas function $F(K,L) = AK^aL^b$.

$$R_{KL} = \frac{\frac{\partial F}{\partial L}}{\frac{\partial F}{\partial K}} = \frac{bAK^aL^{b-1}}{aAK^{a-1}L^b} = \frac{b}{a}\cdot\frac{K}{L}$$

so $\frac{K}{L} = \frac{a}{b}R_{KL}$

$$\sigma_{KL} = El_{R_{KL}}\left(\frac{K}{L}\right) = \frac{R_{KL}}{\frac{K}{L}}\cdot\frac{d\frac{K}{L}}{dR_{KL}} = \frac{R_{KL}}{\frac{a}{b}R_{KL}}\cdot\frac{a}{b} = 1$$