Partial Elasticities

We have defined elasticity for a single-variable function, similarly we can define partial elasticities for a multivariable function.

If $z = F(x, y)$, we define the partial elasticity of z w.r.t. x to be

$El_x z = \frac{x}{z}\frac{\partial z}{\partial x}$, and define the partial elasticity of z w.r.t. y to be $El_y z = \frac{y}{z}\frac{\partial z}{\partial y}$.

The $El_x z$ is approximately the percentage change in z caused by a 1% increase in x, when y is held constant.

The $El_y z$ is approximately the percentage change in z caused by a 1% increase in y. when x is held constant

Example. $D = Ax^a y^b$. Find partial elasticities (A is a constant)

$$El_x D = \frac{x}{D}\frac{\partial D}{\partial x} = \frac{x}{Ax^a y^b}\cdot(A a x^{a-1} y^b) = a.$$

$$El_y D = \frac{y}{D}\frac{\partial D}{\partial y} = \frac{y}{Ax^a y^b}(Abx^a y^{b-1}) = b$$

Example. The demand D_1 for potatoes in U.S. during 1927-1941 was $D_1 = Ap^{-0.28}m^{0.34}$, where p is the price for potatoes and m is the average income. The demand D_2 for apples was $D_2 = Bq^{-1.27}m^{1.32}$

Then $El_p D_1 = -0.28$, $El_m D_1 = 0.34$

$El_q D_2 = -1.27$, $El_m D_2 = 1.32$

The demand in apples is more sensitive w.r.t. both price and income, which agreed with that period of history that potatoes were more essential than apples during that time.

General Multi-variable Differentiation

Partial Derivatives in n Variables:

If $z = f(x_1, x_2, \dots, x_n)$ is a function with n variables, we can define the partial derivatives $\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \dots, \frac{\partial f}{\partial x_n}$ in a same manner as those of functions with two variables.

And the computation follows the same rule: $\frac{\partial f}{\partial x_j}$ is to take derivative with respect to x_j, while regarding other variables as constants.

Example. $f(x, y, z) = x^2 + 2y^2 + xyz$

then $\frac{\partial f}{\partial x} = 2x + yz$, $\frac{\partial f}{\partial y} = 4y + xz$, $\frac{\partial f}{\partial z} = xy$

We can also take higher order partial derivatives:

The second-order partial derivatives are defined to be

$$\frac{\partial^2 f}{\partial x_j \partial x_i} = \frac{\partial}{\partial x_j}\left(\frac{\partial f}{\partial x_i}\right)$$

Example Continue with the previous example.

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = 2, \quad \frac{\partial^2 f}{\partial x \partial y} = z, \quad \frac{\partial^2 f}{\partial x \partial z} = y$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = z, \quad \frac{\partial^2 f}{\partial y^2} = 4 \quad \frac{\partial^2 f}{\partial y \partial z} = x$$

$$\frac{\partial^2 f}{\partial z \partial x} = \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial x}\right) = y, \quad \frac{\partial^2 f}{\partial z \partial y} = x \quad \frac{\partial^2 f}{\partial z^2} = 0$$

Observe that in the previous example,

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}, \quad \frac{\partial^2 f}{\partial x \partial z} = \frac{\partial^2 f}{\partial z \partial x}, \quad \frac{\partial^2 f}{\partial y \partial z} = \frac{\partial^2 f}{\partial z \partial y}.$$

In fact, this is not a coincidence, but it follows from a theorem:

Young's Theorem. Suppose that all the m-th order partial derivatives of $f(x_1, x_2, \dots, x_n)$ are continuous. If any two of them involve differentiating w.r.t. each of the variables the same number of times, then they are necessarily equal.

In particular, when $f(x_1, \dots, x_n)$ has continuous second-order derivatives, we have

$$\frac{\partial^2 f}{\partial x_j \partial x_i} = \frac{\partial^2 f}{\partial x_i \partial x_j}.$$

Remark. There are cases when the Young's Theorem cannot work, if the second-order derivatives are not continuous

An example is given by

$$f(x,y) = \begin{cases} \dfrac{xy(x^2-y^2)}{x^2+y^2} & \text{when } (x,y) \neq (0,0) \\ 0 & \text{when } (x,y) = (0,0) \end{cases}$$

After calculation, we find

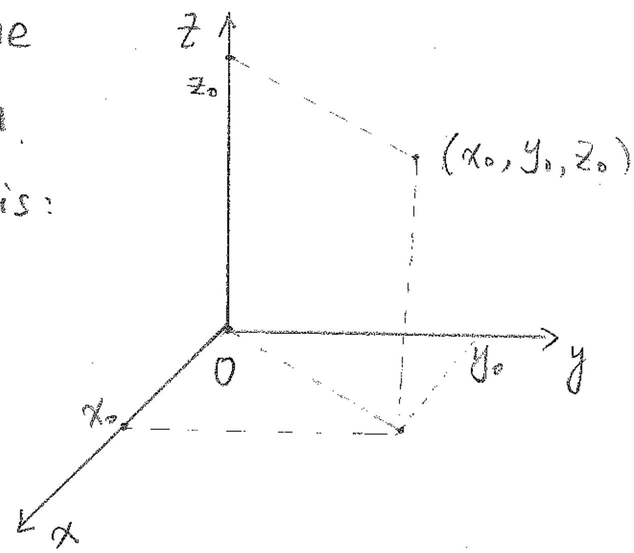
$$\frac{\partial^2 f}{\partial x \partial y} = 1 \quad \text{while} \quad \frac{\partial^2 f}{\partial y \partial x} = -1$$

Geometry and Two-Variables

We can represent a single-variable function by a graph on the coordinate plane. Similarly, for functions of two variables, we can also represent them geometrically.

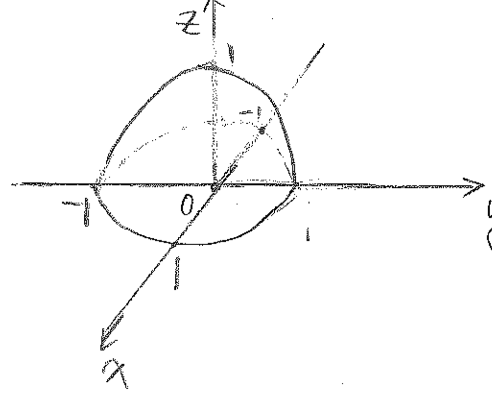
Consider a 3-dimensional space. We choose a horizontal plane, and build an x-y coordinate system on this plane. We take the line passing through the origin, perpendicular to the plane, and pointing upward. we identify this line with a copy of the real-axis, and call it the z-axis. Now each point in the 3-dimensional space can be represented uniquely as three ordered numbers (x, y, z). The height of the point w.r.t. the x-y plane is the z-coordinate, and the coordinates of its projection on the x-y plane gives the x and y coordinates.

The above construction gives the 3-dimensional coordinate system and it's often drawn like this:

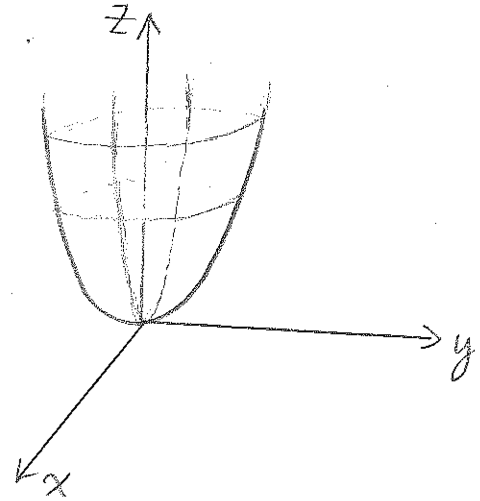


The graph of a function of two variables $f(x, y)$ is the set of points $(x, y, f(x, y))$ in the 3-dimensional coordinate system, it's projection on the x-y plane is the domain of f, and it's projection on the z-coordinate is the range of f.

Example. The graph of $z = f(x, y) = \sqrt{1-x^2-y^2}$ is:

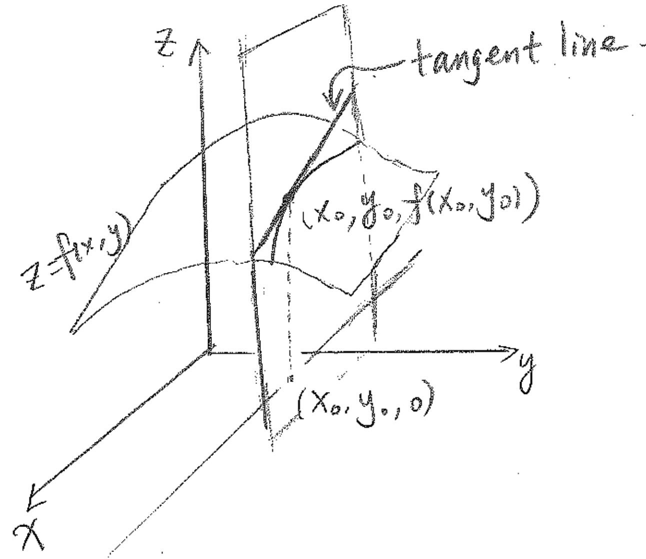


Example. The graph of $z = f(x, y) = x^2+y^2$ is:

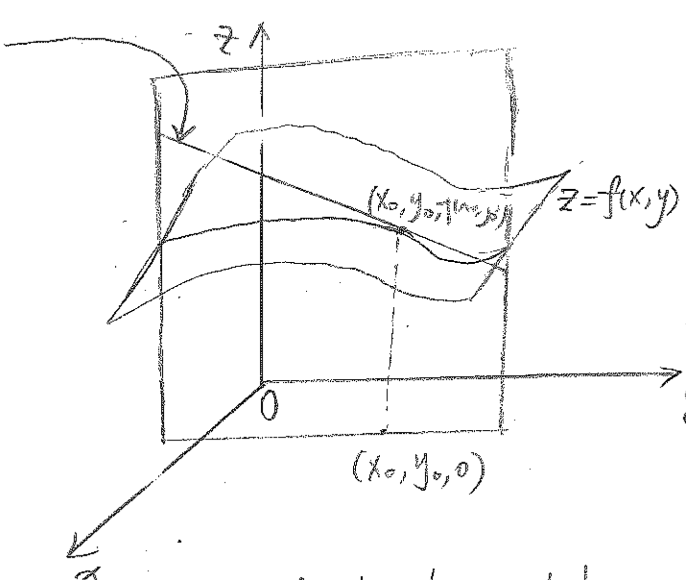


Interpretation of Partial derivatives in geometry:

If $z = f(x, y)$ is a function of two variables, it has corresponding graph. Now at a point $(x_0, y_0, f(x_0, y_0))$ on this graph, we draw the plane parallel to x-z plane and passing through $(x_0, y_0, f(x_0, y_0))$. This plane will intersect the graph at a curve, which consists of points of the form $(x, y_0, f(x, y_0))$. Then the slope of the intersection curve at $(x_0, y_0, f(x_0, y_0))$ is the partial derivative $\frac{\partial f}{\partial x}(x_0, y_0)$. The other partial derivative $\frac{\partial f}{\partial y}(x_0, y_0)$ can be similarly achieved, but we need the plane to be parallel to y-z plane instead.



slope of the tangent line is $\frac{\partial f}{\partial x}(x_0, y_0)$

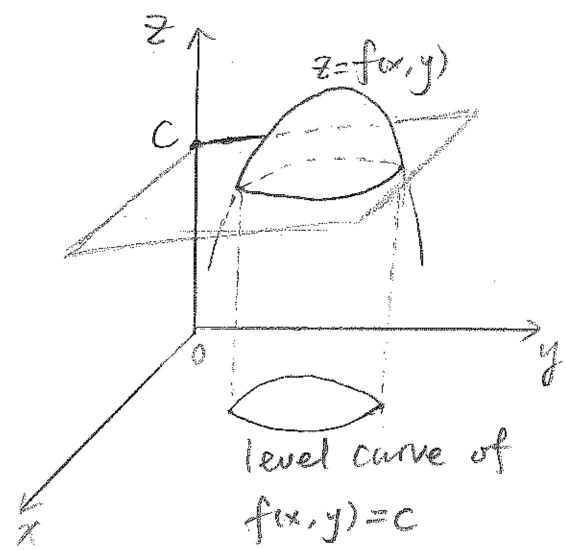


slope of the tangent line is $\frac{\partial f}{\partial y}(x_0, y_0)$

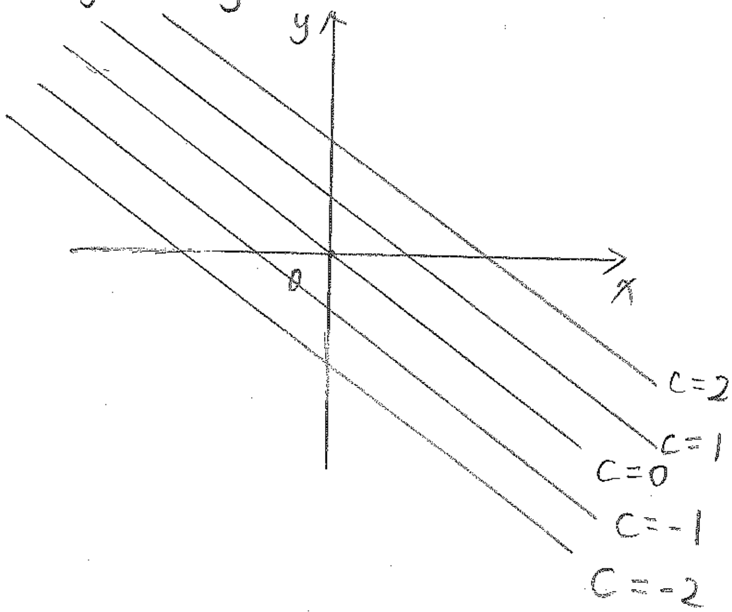
Level curve

Sometimes drawing graphs of 3-dimensions is not an easy job, so people use another method, which gives a representation of the function $z=f(x,y)$ on the x-y plane. What they draw in this case are called the level curves.

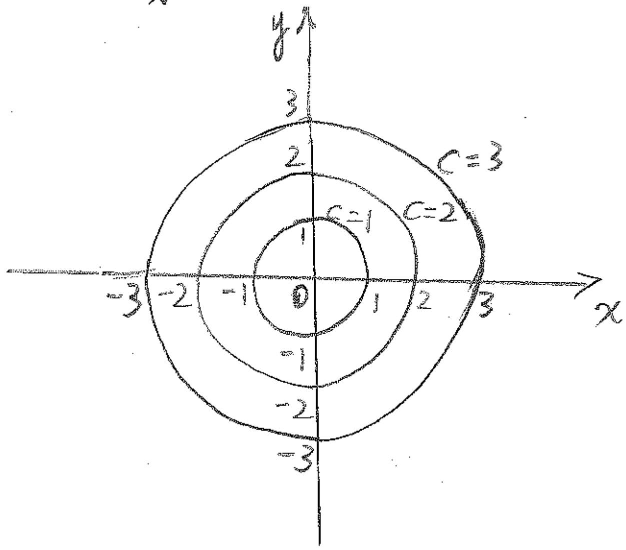
A level curve is the set of points (x,y) on the x-y coordinate plane such that $f(x,y)=c$ for some given constant c. It's the projection of the points on the graph of the function $z=f(x,y)$ whose z-coordinate is c on to the x-y plane.



Example. $z = f(x,y) = x+y$, the level curves are $x+y=c$.



Example. $z = f(x,y) = \sqrt{x^2+y^2}$, the level curves are $\sqrt{x^2+y^2} = c$.



Remark. For more beautifully-drawn pictures and examples, please read the textbook.

The level curves can also tell us about partial derivatives. That is, if along a direction, the density of level curves is high, then it means the function changes quickly in this direction, so it's steeper, i.e. the partial derivative has a larger absolute value.