Chain Rule for Multivariables

Consider a function $z = F(x, y)$. If x and y are both functions of t, say $x = f(t)$ and $y = g(t)$, then z is also a function of t: $z = F(x, y) = F(f(t), g(t))$. So we can talk about the derivative $\frac{dz}{dt}$, which is computable by a Chain Rule.

Theorem. When $z = F(x, y)$ with $x = f(t)$, $y = g(t)$, then:

$$\frac{dz}{dt} = \frac{\partial F}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial F}{\partial y} \cdot \frac{dy}{dt}$$

Example. Find $\frac{dz}{dt}$ when $z = F(x, y) = x^2 + y^3$, with $x = t^2$, $y = 2t$.

$$\frac{dz}{dt} = \frac{\partial}{\partial x}(x^2 + y^3) \cdot (t^2)' + \frac{\partial}{\partial y}(x^2 + y^3) \cdot (2t)'$$

$$= 2x \cdot 2t + 3y^2 \cdot 2$$

$$= 2 \cdot t^2 \cdot 2t + 3(2t)^2 \cdot 2$$

$$= 4t^3 + 24t^2$$

A remark is that we can use another method to do it:

$$z = x^2 + y^3 = (t^2)^2 + (2t)^3 = t^4 + 8t^3$$

then $\frac{dz}{dt} = z' = 4t^3 + 24t^2$

We see we obtain the same result.

There are typical applications of this chain Rule in economics, we are going to see some examples of this type.

Example. D is the demand for a commodity, D is a function of price p and income m, $D = D(p, m)$. Suppose price p and income m vary continuously with time t, so that $p = p(t)$ and $m = m(t)$. Then demand can be determined as a function $D = D(p(t), m(t))$ of t.

Find an expression for $\frac{\dot{D}}{D}$, the relative rate of growth of D. (where $\dot{D} = \frac{dD}{dt}$)

$$\dot{D} = \frac{dD}{dt} = \frac{\partial D}{\partial p} \cdot \frac{dp}{\partial t} + \frac{\partial D}{\partial m} \cdot \frac{dm}{dt}$$

$$\begin{aligned}
\frac{\dot{D}}{D} &= \frac{1}{D} \cdot \frac{\partial D}{\partial p} \cdot \frac{dp}{dt} + \frac{1}{D} \cdot \frac{\partial D}{\partial m} \cdot \frac{dm}{dt} \\
&= \frac{p}{D} \cdot \frac{\partial D}{\partial p} \cdot \frac{\frac{dp}{dt}}{p} + \frac{m}{D} \cdot \frac{\partial D}{\partial m} \cdot \frac{\frac{dm}{dt}}{m} \\
&= (El_p D) \cdot \frac{\dot{p}}{p} + (El_m D) \cdot \frac{\dot{m}}{m}
\end{aligned}$$

So we see the relative rate of change is found by multiplying the relative rate of change of price and income by their respective elasticities, then adding.

Now let's consider a more complicated case:

$z = F(x, y)$, and both x and y are function of t and s.

i.e. $x = f(t, s)$, $y = g(t, s)$, then z is a function of t and s: $z = F(f(t,s), g(t,s))$. We have a way to compute the partial derivatives.

Theorem (Chain Rule)

If $z = F(x, y)$ with $x = f(t, s)$ and $y = g(t, s)$, then

(a) $\frac{\partial z}{\partial t} = \frac{\partial F}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial F}{\partial y} \cdot \frac{\partial y}{\partial t}$

(b) $\frac{\partial z}{\partial s} = \frac{\partial F}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial F}{\partial y} \cdot \frac{\partial y}{\partial s}$

Example. Find $\frac{\partial z}{\partial t}(1,0)$ when $z = F(x, y) = e^{x^2} + y^2 e^{xy}$, with $x = 2t + 3s$ and $y = t^2 s^3$.

$$\frac{\partial z}{\partial t} = \frac{\partial F}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial F}{\partial y} \cdot \frac{\partial y}{\partial t}$$

$$= (2xe^{x^2} + y^3 e^{xy}) \cdot 2 + (2ye^{xy} + y^2 x e^{xy}) \cdot 2ts^3$$

when $t = 1, s = 0$. $x = 2 \times 1 + 3 \times 0 = 2$. $y = 1^2 \times 0^3 = 0$

so $\frac{\partial z}{\partial t}(1,0) = (2 \times 2 \times e^{2^2}) \times 2 = 8e^4$

Example. Find $\frac{\partial z}{\partial t}$ and $\frac{\partial z}{\partial s}$ when $z = F(x, y) = x^2 + 2y^2$, with $x = t - s^2$ and $y = ts$.

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial t}$$

$$= 2x \cdot 1 + 4y \cdot s$$

$$= 2x + 4ys$$

$$= 2 \cdot (t - s^2) + 4(ts)s$$

$$= 2t - 2s^2 + 4ts^2$$

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial s}$$

$$= 2x \cdot (-2s) + 4y \cdot t$$

$$= -4xs + 4yt$$

$$= -4(t - s^2)s + 4(ts)t$$

$$= -4ts + 4s^3 + 4t^2 s$$

General Chain Rule:

Suppose $z = F(x_1, x_2, \ldots, x_n)$, with $x_1 = f_1(t_1, \ldots, t_m), \ldots, x_n = f_n(t_1, \ldots, t_m)$

Then z is a function of $t_1, \ldots, t_n$, with

$$\frac{\partial z}{\partial t_j} = \frac{\partial z}{\partial x_1} \cdot \frac{\partial x_1}{\partial t_j} + \frac{\partial z}{\partial x_2} \cdot \frac{\partial x_2}{\partial t_j} + \cdots + \frac{\partial z}{\partial x_n} \cdot \frac{\partial x_n}{\partial t_j}, \quad j = 1, 2, \ldots, m$$

Example. An agricultural production function $Y = F(K, L, T)$.

Y is the size of harvest, K is capital invested, L is labor, T is the area of agricultural land used to grow the crop.

Suppose K, L, T are functions of time t, then

$$\frac{dY}{dt} = \frac{\partial Y}{\partial K} \cdot \frac{dK}{dt} + \frac{\partial Y}{\partial L} \cdot \frac{dL}{dt} + \frac{\partial Y}{\partial T} \cdot \frac{dT}{dt}$$

Sometimes the production satisfies the Cobb-Douglas function $Y = F(K, L, T) = AK^a L^b T^c$, where A, a, b, c are constants

Then
$$\frac{dY}{dt} = \frac{\partial Y}{\partial K} \cdot \frac{dK}{dt} + \frac{\partial Y}{\partial L} \cdot \frac{dL}{dt} + \frac{\partial Y}{\partial T} \cdot \frac{dT}{dt}$$
$$= AaK^{a-1} L^b T^c \cdot \frac{dK}{dt} + AbK^a L^{b-1} T^c \cdot \frac{dL}{dt} + AcK^a L^b T^{c-1} \frac{dT}{dt}$$

We denote all derivatives with respect to time by dot,

$$\dot{Y} = AaK^{a-1} L^b T^c \cdot \dot{K} + AbK^a L^{b-1} T^c \dot{L} + AcK^a L^b T^{c-1} \cdot \dot{T}$$

$$\frac{\dot{Y}}{Y} = \frac{AaK^{a-1} L^b T^c}{AK^a L^b T^c} \dot{K} + \frac{AbK^a L^{b-1} T^c}{AK^a L^b T^c} \cdot \dot{L} + \frac{AcK^a L^b T^{c-1}}{AK^a L^b T^c} \cdot \dot{T}$$
$$= a \frac{\dot{K}}{K} + b \frac{\dot{L}}{L} + c \frac{\dot{T}}{T}$$

We see the relative rate of change of the output is a weighted sum of the relative rate of change of the three factors

Example. $w = x^2 + y^2 + z^2$, with $x = \sqrt{t+s}$, $y = e^{ts}$, $z = s^3$.

Find $\frac{\partial w}{\partial t}$ and $\frac{\partial w}{\partial s}$:

$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial t} + \frac{\partial w}{\partial z}\cdot\frac{\partial z}{\partial t}$$

$$= 2x\cdot\frac{1}{2}(t+s)^{-\frac{1}{2}} + 2y\cdot(se^{ts}) + 2z\cdot 0$$

$$= x(t+s)^{-\frac{1}{2}} + 2yse^{ts}$$

$$= (t+s)^{\frac{1}{2}}\cdot(t+s)^{-\frac{1}{2}} + 2e^{ts}se^{ts}$$

$$= 1 + 2se^{2ts}$$

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\cdot\frac{\partial z}{\partial s}$$

$$= 2x\cdot\frac{1}{2}(t+s)^{-\frac{1}{2}} + 2y\cdot te^{ts} + 2z\cdot 3s^2$$

$$= x(t+s)^{-\frac{1}{2}} + 2yte^{ts} + 6zs^2$$

$$= (t+s)^{\frac{1}{2}}(t+s)^{-\frac{1}{2}} + 2e^{ts}te^{ts} + 6s^3\cdot s^2$$

$$= 1 + 2te^{2ts} + 6s^5$$