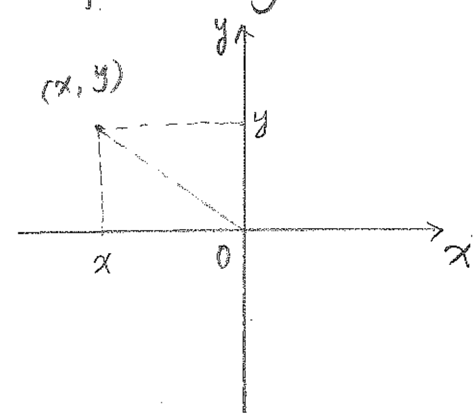
Multi-Variable Functions

Function of two Variables:

A function of two variables x, y with domain D is a rule that assigns a specified number $f(x,y)$ to each point (x,y) in D

Example. $z = f(x,y) = \sqrt{x^2+y^2}$

This function assigns to (x,y) the distance from (x,y) to origin



Example. Ohm's Law

$I = \frac{V}{R}$ The current through a conductor between two points is the quotient of the voltage across the two points and the resistance between the two points

Example. Profit is the difference of revenue and cost:

$P = R - C$

We assume, unless otherwise stated, that the domain of a function defined by a formula is the largest domain in which the formula gives a meaningful and unique value.

Example. $f(x,y) = \frac{1}{x} + \sqrt{y-1}$

We need $x \neq 0$ and $y-1 \geq 0$, i.e $x \neq 0$ and $y \geq 1$

So the domain is $\{(x,y) \in \mathbb{R}^2 \mid x \neq 0 \text{ and } y \geq 1\}$

Example. $f(x) = \ln(x^2+y^2)$

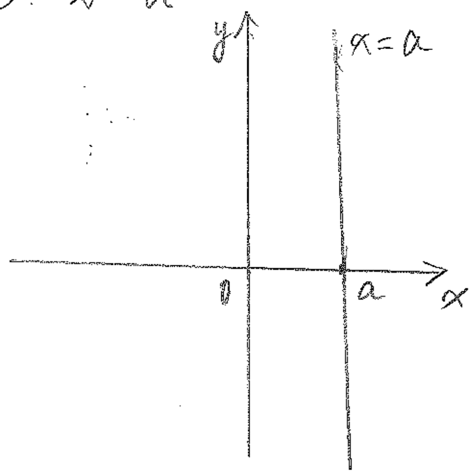
we need $x^2+y^2>0$, and this holds unless $(x,y)=(0,0)$.

so the domain is $\{(x,y)\in\mathbb{R}^2 \mid (x,y)\neq(0,0)\}$

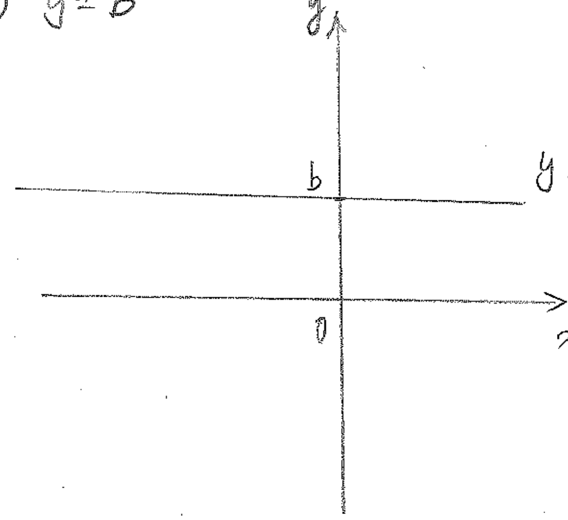
Sometimes we can draw the domain of $z=f(x,y)$ on the coordinate plane, and usually we first find the boundary of the domain, which separates the plane into several regions, and then determine the regions.

Let's first have a brief review of the graphs of some popular equations:

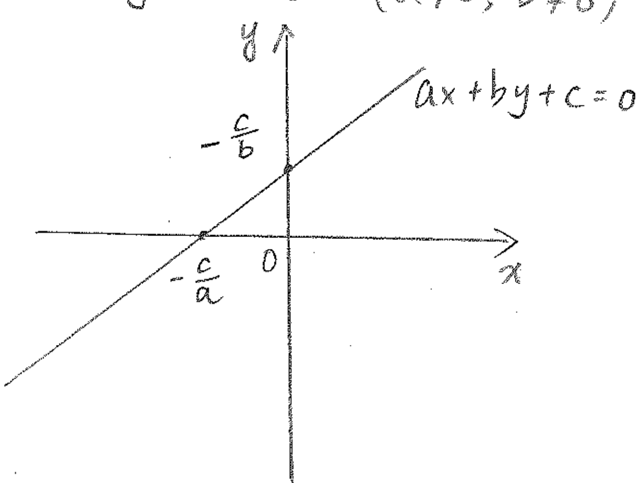
① $x=a$



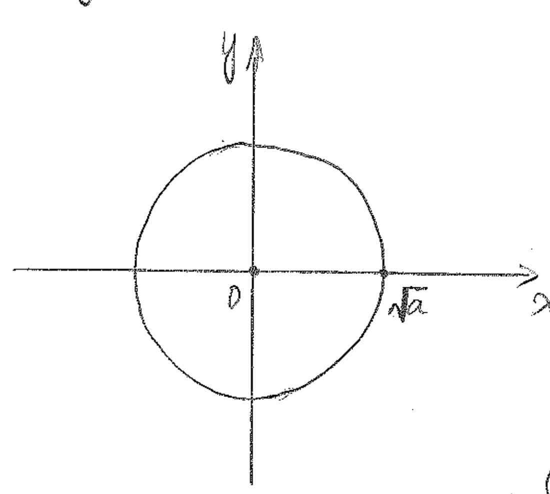
② $y=b$



③ $ax+by+c=0$. $(a\neq0,\ b\neq0)$



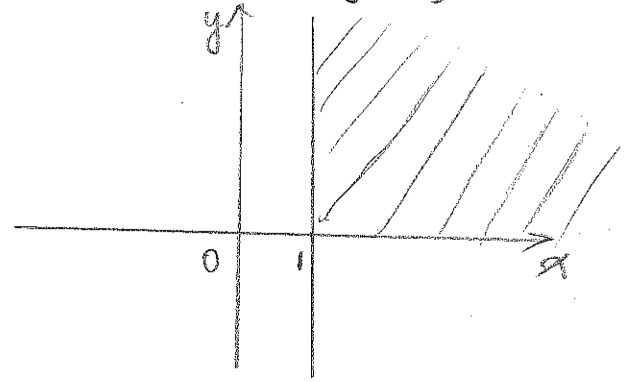
④ $x^2+y^2=a$



Example. Determine the domain of the function $f(x,y) = \sqrt{x-1} + \sqrt{y}$, then draw the sets in the x-y plane

We need $x-1 \geq 0$ and $y \geq 0$, i.e. $x \geq 1$ and $y \geq 0$

So the domain is $\{(x,y) \in \mathbb{R}^2 \mid x \geq 1 \text{ and } y \geq 0\}$



Example Determine the domain of $f(x,y) = \dfrac{\sqrt{9-(x^2+y^2)}}{\sqrt{x^2+y^2-4}}$, then draw the sets in the x-y plane:

We need $9-(x^2+y^2) \geq 0$ and

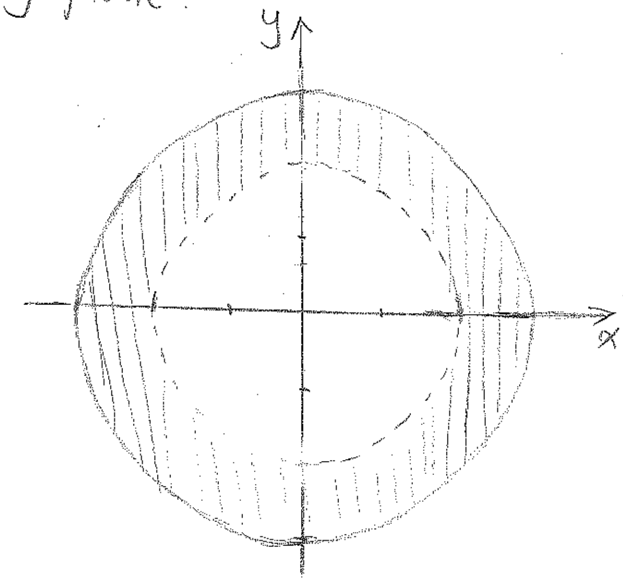
$x^2+y^2-4 > 0$

i.e. $x^2+y^2 \leq 9$ and

$x^2+y^2 > 4$

So the domain is

$\{(x,y) \in \mathbb{R}^2 \mid 4 < x^2+y^2 \leq 9\}$



Partial Derivatives

Recall that for a single-variable function $f(x)$, its derivative measures the rate of change of $y=f(x)$ at x.

For a multi-variable function, we can have an analogue of that, but we need to specify the rate of change is with respect to which variable, since we have more than one variables.

If $z=f(x,y)$ is a function with two variables, we define:

$$\frac{\partial z}{\partial x}=\frac{\partial f}{\partial x}=f_1'(x,y)=\lim_{h\to 0}\frac{f(x+h,y)-f(x,y)}{h}$$

$$\frac{\partial z}{\partial y}=\frac{\partial f}{\partial y}=f_2'(x,y)=\lim_{h\to 0}\frac{f(x,y+h)-f(x,y)}{h}$$

In other words, $\frac{\partial f}{\partial x}$ is to take derivative of f with respect to x, considering y as a constant; $\frac{\partial f}{\partial y}$ is to take derivative of f with respect to y, considering x as a constant.

Example. $f(x,y)=x^3e^{y^2}$, evaluate $\frac{\partial f}{\partial x}(1,0)$ and $\frac{\partial f}{\partial y}(1,0)$

$$\frac{\partial f}{\partial x}=3x^2e^{y^2} \qquad \frac{\partial f}{\partial y}=x^3\cdot 2ye^{y^2}$$

So $\frac{\partial f}{\partial x}(1,0)=3$, $\quad \frac{\partial f}{\partial y}=0$

Example. $f(x,y)=\ln(x^2+y^2)$ find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$.

$$\frac{\partial f}{\partial x}=\frac{2x}{x^2+y^2}, \quad \frac{\partial f}{\partial y}=\frac{2y}{x^2+y^2}$$

How do we interpret partial derivative?

$$\frac{\partial f}{\partial x}(x,y) = \lim_{h \to 0} \frac{f(x+h,y)-f(x,y)}{h}$$

so when h is close to 0.

$$\frac{\partial f}{\partial x}(x,y) \approx \frac{f(x+h,y)-f(x,y)}{h}$$

$$f(x+h,y)-f(x,y) \approx \frac{\partial f}{\partial x}(x,y)\cdot h$$

so $\frac{\partial f}{\partial x}(x,y)$ helps to estimate the change of the function if x is increased by h, and y is fixed.

Example. A factory is producing two brands of commodities, A and B. If x units of Brand A and y units of Brand B is produced, the cost function is

$$C(x,y) = x^{\frac{1}{2}} + y^{\frac{1}{3}}$$

Currently the factory is producing 100 units of A and 125 units of B per day.

$$\frac{\partial C}{\partial x}(x,y) = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$$

The marginal cost at this production with respect to Brand A is $\frac{\partial C}{\partial x}(100,125) = \frac{1}{2\sqrt{100}} = \frac{1}{20}$

Second-Order Partial Derivative:

We can take the partial derivative of a partical derivative to obtain a second order partial derivative:

Define $\frac{\partial^2 f}{\partial x^2} = f''_{11} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)$ $\quad \frac{\partial^2 f}{\partial x \partial y} = f''_{21} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$

$\frac{\partial^2 f}{\partial y \partial x} = f''_{12} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$ $\quad \frac{\partial^2 f}{\partial y^2} = f''_{22} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)$

Example: Find all the second order partial derivatives of the function $f(x,y) = x^3 y + x^2 y^2 + x + y^2$

$\frac{\partial f}{\partial x} = 3x^2 y + 2xy^2 + 1$, $\frac{\partial f}{\partial y} = x^3 + 2x^2 y + 2y$

so: $\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = 6xy + 2y^2$ $\quad \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = 3x^2 + 4xy$

$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = 3x^2 + 4xy.$ $\quad \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial}{\partial y}\right) = 2x^2 + 2$

Example: Find all the second order partial derivatives of the function $f(x,y) = x \ln y$

$\frac{\partial f}{\partial x} = \ln y$ $\quad \frac{\partial f}{\partial y} = \frac{x}{y}$

So $\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}(\ln y) = 0$ $\quad \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}\left(\frac{x}{y}\right) = \frac{1}{y}$

$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}(\ln y) = \frac{1}{y}$ $\quad \frac{\partial^2 f}{\partial^2 y} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{x}{y}\right) = -\frac{x}{y^2}$