Extreme Points & Extreme Values

f is a function with domain D, we define:

$c \in D$ is a maximum point for $f \iff f(x) \le f(c)$ for all $x \in D$

$c \in D$ is a minimum point for $f \iff f(x) \ge f(c)$ for all $x \in D$

We say c is an extreme point if it's a maximum point or minimum point.

If c is a maximum point of f, we call $f(c)$ the maximum value of f on D.

If c is a minimum point of f, we call $f(c)$ the minimum value of f on D.

We say $f(c)$ is an extreme value if it's a maximum value or a minimal value of f.

$c \in D$ is a strict maximum point for $f \iff f(x) < f(c)$ for all $x \in D$, $x \neq c$

$c \in D$ is a strict minimum point for $f \iff f(x) > f(c)$ for all $x \in D$, $x \neq c$

Example. $f(x) = 1 - x^2$

We know $x^2 \geq 0$ and $x^2 = 0$ if and only if $x = 0$

so $-x^2 \le 0$, and $-x^2 = 0$ if and only if $x = 0$

so $f(x) = 1 - x^2 \le 1$, and $1 - x^2 = 1$ if and only if $x = 0$

we see $x = 0$ is a maximum point of f, with the corresponding maximum value $f(0) = 1$

Since $\lim\limits_{x \to +\infty} f(x) = -\infty$, the function has no minimum

Example. $f(x) = \sqrt{x-1} + 30$

$\sqrt{x-1} \geq 0$, and $\sqrt{x-1} = 0$ if and only if $x = 1$

So $f(x) = \sqrt{x-1} + 30 \geq 30$, and $f(x) = 30$ if and only if $x = 1$

we see $x = 1$ is a minimum point of f, with the corresponding minimal value $f(1) = 30$.

Since $\lim_{x \to +\infty} f(x) = +\infty$, the function has no maximum

In general, finding extreme points is not an easy job, if we just work by the method in the above examples.

With the help of calculus, we have a much easier way to find extreme points

Theorem. (Necessary First-Order Condition)

Suppose f is differentiable function defined on an interval I, and c is an interior point of I. If c is an extreme point, then $f'(c) = 0$

Remark. The above theorem offers a necessary condition for c to be an extreme point, but the condition in general is not sufficient, that is, $f'(c) = 0$ doesn't always imply c is an extreme point

In some cases, with some extra conditions, we can conclude a point is maximal or minimal:

Theorem (First-Derivative Test for Maximum/Minimum)

If $f'(x) \geq 0$ for all $x \leq c$ and $f'(x) \leq 0$ for all $x \geq c$, then $x = c$ is a maximum point for f.

If $f'(x) \leq 0$ for all $x \leq c$ and $f'(x) \geq 0$ for all $x \geq c$, then $x = c$ is a minimum point for f

Example. $f(x) = x^2$

$$f'(x) = 2x \quad \text{and} \quad \begin{cases} f'(x) \leq 0 & \text{for all } x \leq 0, \\ f'(x) \geq 0 & \text{for all } x \geq 0 \end{cases}$$

So by the above Theorem, $x = 0$ is a minimum point for f

Example. $f(x) = e^{-x^2+2x}$

$$f'(x) = (-2x+2)e^{-x^2+2x} = 2(1-x)e^{-x^2+2x}$$

so $\begin{cases} f'(x) \geq 0 & \text{for all } x \leq 1 \\ f'(x) \leq 0 & \text{for all } x \geq 1 \end{cases}$

So $x = 1$ is a maximum point.

If we know f is convex or concave function, then it is easier to find extreme points.

Theorem (Extreme Point for Convex/Concave Functions)
If f is a convex function in an interval I, and $f'(c)=0$, then c is a minimum point for f in I.
If f is a concave function in an interval I, and $f'(c)=0$, then c is a maximum point for f in I.

Example. $f(x) = e^{x-1} - x$

$f'(x) = e^{x-1} - 1, \quad f''(x) = e^{x-1}$

We see $f''(x) > 0$ for all $x \in \mathbb{R}$, so f is convex on $\mathbb{R}$

$f'(x) = e^{x-1} - 1 = 0 \Leftrightarrow e^{x-1} = 1 \Leftrightarrow x - 1 = 0 \Leftrightarrow x = 1$

So $x=1$ is a minimum point of f.

the corresponding minimum value is $f(0) = e^{1-1} - 1 = 0$

Example. Show that $\ln x \leq x - 1$ for all $x > 0$.

Define $f(x) = \ln x - (x-1)$

then $f'(x) = \frac{1}{x} - 1, \quad f''(x) = -\frac{1}{x^2}$

We see $f''(x) < 0$ for all $x \in (0, +\infty)$

and $f'(x) = \frac{1}{x} - 1 = 0 \Leftrightarrow x = 1$

So $x=1$ is a maximum point, with maximum value $f(1) = 0$

So $f(x) = \ln x - (x-1) \leq f(1) = 0 \Rightarrow \ln x < x - 1$

Example. If an object is thrown upward with initial velocity v_0 at height h_0, and it's given that at time t, its distance to the ground is

$$h = h_0 + v_0 t - \frac{1}{2} g t^2$$

Find the time when it's at highest point and the corresponding height.

$h' = v_0 - gt$, so when $t \leq \frac{v_0}{g}$, $h' \geq 0$ and

$t \geq \frac{v_0}{g}$, $h' \leq 0$

we conclude $t = \frac{v_0}{g}$ is a maximum point of h,

and the corresponding maximum value is

$$h = h_0 + v_0 \cdot \frac{v_0}{g} - \frac{1}{2} g \cdot \left(\frac{v_0}{g}\right)^2 = h_0 + \frac{v_0^2}{2g}$$

Remark. We can also argue by $h'' = -g \Rightarrow h$ is concave

and $f'\left(\frac{v_0}{g}\right) = 0$ to conclude $t = \frac{v_0}{g}$ is a maximum point.

Example. The total cost of producing Q units of commodity is

$C(Q) = aQ^2 + bQ + c \quad (Q > 0)$, where a, b, c are positive constants. Show that the average cost function

$$A(Q) = \frac{C(Q)}{Q} = aQ + b + \frac{c}{Q}$$

has a minimum at $Q^* = \sqrt{\frac{c}{a}}$

Since $A(Q) = aQ + b + \frac{c}{Q}$, $A'(Q) = a - \frac{c}{Q^2}$ and $A''(Q) = \frac{2c}{Q^3}$

when $Q > 0$, $A''(Q) = \frac{2c}{Q^3} > 0$. So $A(Q)$ is convex.

$$A'(Q^*) = A'\left(\sqrt{\tfrac{c}{a}}\right) = a - \frac{c}{\left(\sqrt{\frac{c}{a}}\right)^2} = 0.$$

So Q^* is a minimum point for $A(Q)$

Now we are going to study the extreme points of a function defined on a closed interval $I = [a, b]$.

The following theorem guarantees the existence of maximum and minimum points in this case:

Theorem: f is a differentiable function over a closed interval $[a, b]$. Then there exists $d \in [a, b]$ where f has minimum, and $c \in [a, b]$ where f has maximum.

So now we know the existence, but the problem is how to find the extreme points. The first step is to look for the candidates, and then choose from them. We need the concept of Local Extreme Points in order to find candidates.

A function f has a local maximum at c, if there exists an interval (α, β) about c such that $f(x) \leq f(c)$ for all $x \in (\alpha, \beta)$.

A function f has a local minimum at c, if there exists an interval (α, β) about c, such that $f(x) \geq f(c)$ for all $x \in (\alpha, \beta)$

We call c to be a critical point if $f'(c)=0$ or $f'(c)$ does not exist.

Theorem. If c is a local extreme point of f and c is an interior point in the domain, then c is a critical point.

Theorem. If c is an extreme point of f, then c is a local extreme point of f.

The above two theorems provide us with a way to find extreme points of f on a closed interval $[a,b]$:

Take $f(a)$, $f(b)$ and all the critical points on (a,b), they're the candidates of extreme points. That is, it suffices to restrict the search of extreme points to these candidates.

Example. Find the maximum and minimum point of

$$f(x) = x^3 - 9x^2 + 24x + 6 \text{ on } [0,5].$$

Step 1. Find critical points:

$$f'(x) = 3x^2 - 18x + 24 = 3(x^2 - 6x + 8) = 3(x-2)(x-4)$$

So the critical points are $x=2$ and $x=4$ (when $f'(x)=0$)

Note both 2 and 4 are in $(0,5)$

Step 2. So the candidates are: $0, 2, 4, 5$

Step 3. $f(0)=6$, $f(2)=26$, $f(4)=22$ $f(5)=26$

So maximum points are 2 and 5 maximum value is 26

minimum points is 0 minimum value is 6

Example. Find the maximum and minimum points of

$$f(x) = x + \frac{1}{x} \quad \text{on } [\tfrac{1}{2}, 3]$$

The critical points are : $f(x)' = 1 - \frac{1}{x^2} = 0 \Rightarrow x = 1$ or $x = -1$

So the critical point in the interior of $[\frac{1}{2}, 3]$ is 1

The candidates of extreme points are : $\frac{1}{2}, 1, 3$

$f(\frac{1}{2}) = \frac{5}{2}$, $f(1) = 2$, $f(3) = \frac{10}{3}$

So $x = 3$ is a maximum point, the maximum value is $\frac{10}{3}$

$x = 1$ is a minimum point, the minimum value is 2

Example Find the maximum and minimum points of

$$f(x) = x^{\frac{1}{3}} \quad \text{on } [-1, 1].$$

$f'(x) = \frac{1}{3} x^{-\frac{2}{3}} = \frac{1}{3\sqrt[3]{x^2}}$, when $x = 0$, $f'(x)$ doesn't exist.

So the critical point is 0

The candidates of extreme points are : $-1, 0, 1$

$f(-1) = (-1)^{\frac{1}{3}} = -1$, $f(0) = 0^{\frac{1}{3}} = 0$, $f(1) = 1^{\frac{1}{3}} = 1$

So $x = -1$ is the minimum point, the minimum value is -1

$x = 1$ is the maximum point, the maximum value is 1

The next question is how to find local extremes?

The idea is that when restricted to very small interval containing a local extreme point, this local extreme point becomes an extreme point on this small interval.

Theorem (First Derivative Test)

Suppose $f'(c)=0$, where f is a differentiable function

(a). If $f'(x) \geq 0$ throughout some interval (a,c) and
$f'(x) \leq 0$ throughout some interval (c,b)
then $x=c$ is a local maximum point of f.

(b). If $f'(x) \leq 0$ throughout some interval (a,c) and
$f'(x) \geq 0$ throughout some interval (c,b)
then $x=c$ is a local minimum point of f

(c). If $f'(x) > 0$ both throughout some intervals (a,c) and (c,b)
or if $f'(x) < 0$ both throughout some intervals (a,c) and (c,b)
then $x=c$ is not a local extreme point of f.

Example: $f(x) = x^2 e^x$

$$f'(x) = 2xe^x + x^2e^x = x(2+x)e^x = 0 \Rightarrow x=0 \text{ or } x=-2.$$

$f'(x)$: >0 on $(-\infty,-2)$, <0 on $(-2,0)$, >0 on $(0,\infty)$ (x-axis with points -2 and 0)

So $x=-2$ is a local maximum point
$x=0$ is a local minimum point

Example. $f(x) = x^3$.

$f'(x) = 3x^2 = 0 \Rightarrow x = 0$ is the critical point.

But when $x<0$, $f'(x)>0$. when $x>0$, also $f'(x)>0$

so $x=0$ is not a local extreme point.

There's no local extreme point for $f(x) = x^3$.

Theorem (Second-Derivative Test)

Let f be a twice differentiable function in an interval I, and let c be an interior point of I. Then:

(a) $f'(c)=0$ & $f''(c)<0 \Rightarrow x=c$ is a strict local maximum point

(b). $f'(c)=0$ & $f''(c)>0 \Rightarrow x=c$ is a strict local minimum point.

Example. Let's do $f(x) = x^2 e^x$ again:

$f'(x) = (x^2+2x)e^x = 0 \Rightarrow x=0$ or $x=-2$

$f''(x) = (2x+2)e^x + (x^2+2x)e^x = (x^2+4x+2)e^x$

$f''(0) = 2 > 0$. So $x=0$ is a local minimum point.

$f''(-2) = -2e^{-2} < 0$. so $x=-2$ is a local maximal point.