Elasticities

If f is a function differentiable at x and $f(x) \neq 0$, we can define the elasticity of f with respect to x as:

$$\mathrm{El}_x f(x) = \frac{x}{f(x)} f'(x)$$

What is the meaning of this definition?

Recall that $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$$\mathrm{El}_x f(x) = \frac{x}{f(x)} f'(x) = \frac{x}{f(x)} \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\frac{f(x+h) - f(x)}{f(x)}}{\frac{h}{x}}$$

h is the amount of change in x. so

$\frac{h}{x}$ is the ratio of change in x.

$f(x+h) - f(x)$ is the amount of change in $f(x)$.

$\frac{f(x+h) - f(x)}{f(x)}$ is the ratio of change in $f(x)$.

When h is small, $\mathrm{El}_x f(x) \approx \frac{\frac{f(x+h) - f(x)}{f(x)}}{\frac{h}{x}}$. so $\mathrm{El}_x f(x)$ can be interpreted as: when x is changed by 1%, $f(x)$ will be changed by $\mathrm{El}_x f(x) \cdot 1\%$

Note that the $\mathrm{El}_x f(x)$ is different from the ratio of change $\frac{f'(x)}{f(x)}$ which we have defined before. The meaning of ratio of change is to tell us when x is increased by 1, $f(x)$ will be changed by $\frac{f'(x)}{f(x)} \times 100\%$

Example. Price Elasticity of Demand.

The demand of certain commodity can be described as a function of its price: $Q = D(p)$ where Q is the demand quantity, p is the price, D is the function.

We are interested in the question: at price p, if the price increases by 1%, what's the change of demand in percentage?

The answer is given by the Elasticity of the function $Q=D(p)$.

For instance, if the quantity demanded of a particular commodity is given by $D(p) = 8000p^{-1.5}$, then the Price Elasticity of Demand is

$$El_p D(p) = \frac{p}{D(p)} \cdot D'(p) = \frac{p}{8000p^{-1.5}} \cdot \left(8000 \times (-1.5p^{-2.5})\right) = -1.5$$

This means the increase of price by 1% will lead to a decrease of demand by around 1.5%.

Note that in the above example, the elasticity is a constant. In this case we say this commodity has constant elasticity.

More terminology on elasticity:

If $|El_x f(x)| > 1$, we say f is elastic at x

If $|El_x f(x)| = 1$, we say f is unit elastic at x

If $|El_x f(x)| < 1$, we say f is inelastic at x

If $|El_x f(x)| = 0$, we say f is perfectly inelastic at x

If $|El_x f(x)| = +\infty$ we say f is perfectly elastic at x

Example. We again let $Q = D(p)$ be the demand of a commodity at price P. The revenue $R(p) = P \cdot Q = pD(p)$

Let's compute the elasticity of $R(p)$ with respect to price:

$$El_p R(P) = \frac{p}{R(p)} R'(p) = \frac{p}{p \cdot D(p)} (pD(p))'$$

$$= \frac{1}{D(p)} \cdot [D(p) + pD'(p)]$$

$$= 1 + \frac{p}{D(p)} D'(p)$$

$$= 1 + El_p D(p)$$

A special case is when $El_p D(p) = -1$. In this case $El_p R(p) = 0$. This means at a price whose demand elasticity of price is -1, a small price change in percentage will have almost no influence on the revenue.

More general, we see if $El_p D(p) < -1$ (elastic case), $El_p R(p) < 0$ so the marginal revenue $R'(p)$ is negative. If $-1 < El_p D(p) < 0$ (inelastic case), $R'(p)$ is positive.

The intuition is that if $El_p D(p) < -1$, the decrease in demand is larger than the increase in price, so the revenue, which is their product, will decrease. Similar argument can be made for the other case.

Elasticity as Logarithmic Derivatives.

Recall the Logarithmic Differentiation: $(\ln f(x))' = \frac{f'(x)}{f(x)}$

So the differential $d\ln f(x) = (\ln f(x))' dx = \frac{f'(x)}{f(x)} dx$

$d\ln x = (\ln x)' dx = \frac{1}{x} dx$

$$El_x f(x) = \frac{x}{f(x)} \cdot f'(x) = \frac{\frac{f'(x)}{f(x)}}{\frac{1}{x}} = \frac{\frac{f'(x)}{f(x)} dx}{\frac{1}{x} dx} = \frac{d\ln f(x)}{d\ln x}$$

So we can also interpret the elasticity as a quotient of the differentials $d\ln f(x)$ and $d\ln x$, which reflects the rate of change of $\ln f(x)$ with respect to $\ln x$

A special case is $y = f(x) = Ax^b$. (A, b are constants)

$$El_x f(x) = \frac{d\ln f(x)}{d\ln x} = \frac{d\ln Ax^b}{d\ln x} = \frac{d(\ln A + b\ln x)}{d\ln x} = \frac{b\, d\ln x}{d\ln x} = b$$

which agrees with the computation

$$El_x f(x) = \frac{x}{f(x)} f'(x) = \frac{x}{Ax^b}(Ax^b)' = \frac{x}{Ax^b} \cdot Abx^{b-1} = b$$

Example. Show that $El_x(fg) = El_x(f) + El_x(g)$

$$El_x(fg) = \frac{x}{f(x)g(x)}(f(x)g(x))' = \frac{x}{f(x)g(x)}(f'(x)g(x) + f(x)g'(x))$$

$$= \frac{x}{f(x)} f'(x) + \frac{x}{g(x)} g'(x)$$

$$= El_x f + El_x g$$