Approximations

Linear Approximation.

We have defined the derivative of a function f at $x=a$ to be the limit $f'(a) = \lim_{h \to 0} \frac{f(a+h)-f(a)}{h}$

This expression of the limit means that as h tends to 0, the value of $\frac{f(a+h)-f(a)}{h}$ tends to $f'(a)$. So when h is close to 0, i.e. $|h|$ is a very small number, $\frac{f(a+h)-f(a)}{h}$ is very close to $f'(a)$. So we can write $f'(a) \approx \frac{f(a+h)-f(a)}{h}$ when h is small.

which implies $f(a+h) - f(a) \approx f'(a)h$.

$$f(a+h) \approx f(a) + f'(a)h$$

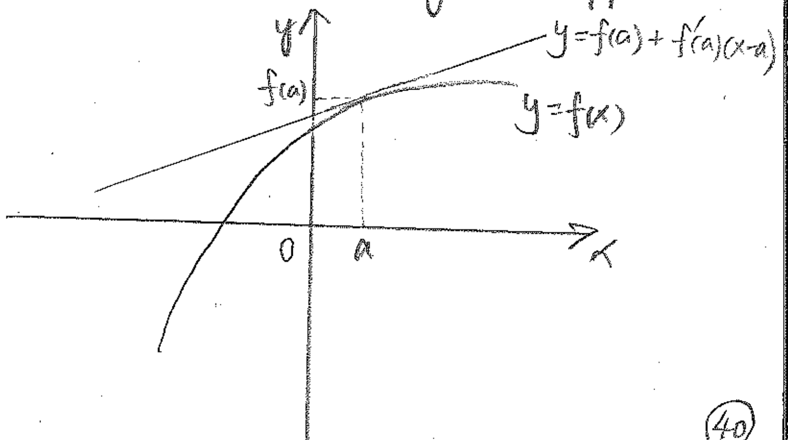
Now let $x = a+h$, since h is close to 0, x is close to a

The above formula becomes:

$$f(x) \approx f(a) + f'(a)(x-a).$$

So we see the function can be approximated by a linear function near $x=a$.

Note that $y = f(a) + f'(a)(x-a)$ is the equation of the tangent line of f at $x=a$. So what we have done is exactly to approximate the function f near $x=a$ by the tangent line of f at $x=a$.



The above method of appoximation is called linear approximation

We obtained the linear approximation to f around $x=a$:

$$f(x) \approx f(a) + f'(a)(x-a) \quad (x \text{ is close to } a).$$

Example. Find the linear approximation to $f(x)=\sqrt{x}$ around $x=1$

$$f'(x) = \frac{1}{2}\frac{1}{\sqrt{x}} \quad \text{so } f(1)=1 \text{ and } f'(1)=\frac{1}{2}$$

So near $x=1$. $f(x) \approx f(1) + f'(1)(x-1) = 1+\frac{1}{2}(x-1)$

How accurate is the approximation?

$f(1.01) = \sqrt{1.01} = 1.00498756\ldots$

$1+\frac{1}{2}(1.01-1) = 1+\frac{1}{2}\times 0.01 = 1.005$

Example. Show that $a^x \approx 1+(\ln a)x$ for x close to 0.

Let $f(x)=a^x$ $\quad f'(x) = (a^x)' = a^x \ln a$.

So. $f(0)=1$. $f'(0) = \ln a$

When x is close to 0, $f(x) \approx f(0) + f'(0)(x-0) = 1+(\ln a)x$

Example. Approximate the value $\ln(1.01)$ without using a calculator.

$f(x)=\ln x$. $\quad f'(x) = \frac{1}{x}$ $\quad f(1) = \ln 1 = 0$. $\quad f'(1)=1$.

so when x is close to 1,

$$f(x) \approx f(1) + f'(1)(x-1) = 0 + x - 1 = x-1$$

$f(1.01) \approx 0.01$

Differential:

f is a differentiable function. Denote dx to be the change of x.
Define $dy = df = f'(x)\,dx$, and call it the differential of $y = f(x)$

Observe than when x changes by dx, the corresponding change in the function value is $f(x+dx) - f(x)$.

Linear approximation tells us that when dx is very small,

$$f(x+dx) \approx f(x) + f'(x)(x+dx-x)$$

i.e. $f(x+dx) - f(x) \approx f'(x)\,dx = df$

So df is the approximation of the change of function values when x changes by dx.

Theorem (Rules for Differentials)

① $d(af+bg) = a\,df + b\,dg$ (a, b are constants)

② $d(fg) = g\,df + f\,dg$

③ $d\left(\frac{f}{g}\right) = \dfrac{g\,df - f\,dg}{g^2}$ $(g \neq 0)$

Example. For a financial investment, the profit y is a function of the input x: $y = f(x)$. If it's given that $f'(10^6) = \frac{1}{5}$, appoximate the increase in the profit if an investor increases input from 10^6 by 1000.

$$f(10^6 + 1000) - f(10^6) \approx df = f'(10^6)\,dx = \frac{1}{5} \times 1000 = 200$$

Higher-Order Approximations

Sometimes the linear approximation doesn't provide us with accurate enough answers, so we need to approximate a function by a polynomial of higher degree.

The approximation of a function f by an n-th degree polynomial near $x = a$ is given by:

$$f(x) \approx f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Example. Approximate e^x by an n-th degree polynomial near $x=0$:

Let $f(x) = e^x$, then $f'(x) = f''(x) = \cdots = f^{(n)}(x) = e^x$

$$\text{so } f(x) = e^x \approx e^0 + \frac{e^0}{1!}(x-0) + \frac{e^0}{2!}(x-0)^2 + \cdots + \frac{e^0}{n!}(x-0)^n$$

$$= 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}$$

When $n=2$, we call the corresponding approximation to be the quadratic approximation.

An intuitive explanation about why the above formula gives an approximation: When two functions are close to each other near a point $x=a$, we expect their derivatives and higher order derivatives are close to each other. So we choose the n-degree polynomial whose first n-order derivatives all equal to those of f at $x=a$.

L'Hôpital's Rule

When $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$, it is usually not easy to tell the limit $\lim_{a\to 0} \frac{f(x)}{g(x)}$. Now by making use of derivatives, we have a new method to help, called the L'Hôpital's Rule.

Theorem. (L'Hôpital's Rule)

f and g are differentiable functions near $x=a$. Suppose $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$. If $g'(x) \neq 0$ for all $x \neq a$ near a, and if $\lim_{x\to a} \frac{f'(x)}{g'(x)} = L$, then $\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)} = L$.

Example. Find $\lim_{x\to 0} \frac{a^x - b^x}{x}$ $\quad (a>0, a\neq 1, b>0, b\neq 1)$

Let $f(x) = a^x - b^x$, $g(x) = x$

then $\lim_{x\to 0} f(x) = \lim_{x\to 0} a^x - \lim_{x\to 0} b^x = a^0 - b^0 = 1$

$\lim_{x\to 0} g(x) = \lim_{x\to 0} x = 0$

So we can apply the L'Hôpital's Rule:

$$\lim_{x\to 0} \frac{a^x - b^x}{x} = \lim_{x\to 0} \frac{f(x)}{g(x)} = \lim_{x\to 0} \frac{f'(x)}{g'(x)} = \lim_{x\to 0} \frac{a^x \ln a - b^x \ln b}{1}$$

$$= \lim_{x\to 0} (a^x \ln a) - \lim_{x\to 0} (b^x \ln b)$$

$$= \ln a - \ln b$$

$$= \ln \frac{a}{b}$$

Example. Find $\lim_{x\to 5} \frac{x^2-2x-15}{x-5}$

Now we can use L'Hospital's Rule since

$\lim_{x\to 5}(x^2-2x-15)=0$ and $\lim_{x\to 5}(x-5)=0$

So $\lim_{x\to 5} \frac{x^2-2x-15}{x-5} = \lim_{x\to 5} \frac{(x^2-2x-15)'}{(x-5)'} = \lim_{x\to 5} \frac{2x-2}{1} = 8$

Example. Sometimes we may need to apply L'Hospital's Rule Repeatedly.

Find $\lim_{x\to 0} \frac{e^x-x-1}{x^2}$

Since $\lim_{x\to 0}(e^x-x-1) = e^0-0-1=0$ and $\lim_{x\to 0} x^2=0$

We apply the L'Hôpital's Rule:

$$\lim_{x\to 0} \frac{e^x-x-1}{x^2} = \lim_{x\to 0} \frac{e^x-1}{2x}$$

Since $\lim_{x\to 0}(e^x-1)=e^0-1=0$. $\lim_{x\to 0} 2x=0$

we apply the L'Hôpital's Rule again:

$$\lim_{x\to 0} \frac{e^x-1}{2x} = \lim_{x\to 0} \frac{e^x}{2} = \frac{1}{2}$$

So $\lim_{x\to 0} \frac{e^x-x-1}{x^2} = \frac{1}{2}$.

Warning: Before applying the L'Hôpital's Rule, you must check the conditions $\lim_{x\to 0} f(x)=0$ and $\lim_{x\to 0} g(x)=0$.

Generalized L'Hôpital's Rule:

The condition " $\lim_{x\to 0} f(x) = 0$ and $\lim_{x\to 0} g(x) = 0$ " can be replaced by " $\lim_{x\to a} f(x) = \pm\infty$ and $\lim_{x\to a} g(x) = \pm\infty$ " in order to apply L'Hôpital's Rule.

Example. Find $\lim_{x\to +\infty} \frac{x}{e^x}$.

Since $\lim_{x\to+\infty}(x) = +\infty$, $\lim_{x\to+\infty} e^x = +\infty$.

Apply the L'Hôpital's Rule, we get

$$\lim_{x\to+\infty} \frac{x}{e^x} = \lim_{x\to+\infty} \frac{1}{e^x} = 0$$

Example. Find $\lim_{x\to+\infty} \frac{\ln x}{-x}$

Since $\lim_{x\to+\infty} \ln x = +\infty$, $\lim_{x\to+\infty}(-x) = -\infty$.

Apply the L'Hôpital's Rule, we get

$$\lim_{x\to+\infty} \frac{\ln x}{-x} = \lim_{x\to+\infty} \frac{\frac{1}{x}}{-1} = \lim_{x\to+\infty} \frac{-1}{x} = 0$$

Example. Find $\lim_{x\to 0} x^2 \ln x^2$.

$\lim_{x\to 0} x^2 = 0$, $\lim_{x\to 0} \ln x^2 = -\infty$.

But $x^2 \ln x^2 = \frac{\ln x^2}{\frac{1}{x^2}}$, $\lim_{x\to 0}\left(\frac{1}{x^2}\right) = +\infty$.

so we can apply the L'Hôpital's Rule:

$$\lim_{x\to 0} x^2 \ln x^2 = \lim_{x\to 0} \frac{\ln x^2}{\frac{1}{x^2}} = \lim_{x\to 0} \frac{2x\cdot\frac{1}{x^2}}{-2\cdot\frac{1}{x^3}} = \lim_{x\to 0}(-x^2) = 0$$