Implicit Differentiation

A variable y is a function of a variable x in general does not imply we can find easily an expression $y=f(x)$. Instead, the relation between x and y may be given by an equation. In such cases, we still can compute the derivative y' by the method in this section.

Example. $y^3+3x^2y=13$. where y is a function of x

It is hard to write $y=f(x)$ for some f.

But we still want to find the rate of change of y with respect to x, i.e. y'.

Method of Implicit Differentiation:

Step 1. Differentiate each side of the equation with respect to x, considering y as a function of x.

Step 2. Solve the resulting equation for y'.

Example. For the previous example. $y^3+3x^2y=13$.

Take derivative with respect to x on both sides.

$$3y^2\cdot y' + 3\cdot 2x\cdot y + 3\cdot x^2\cdot y' = 0$$

so $(3x^2+3y^2)y' = -6xy$

$$y' = \frac{-2xy}{x^2+y^2}$$

Each equation of x, y, $E(x,y)=0$, corresponds to a graph on the coordinate plane consisting of (a,b) such that $E(a,b)=0$.

For example, the equation $x^2+y^2=1$ corresponds to the collection of points (a,b) on the Cartician Coordinate plane such that $a^2+b^2=1$, which is the unit circle centered at origin.

The graph obtained in this way is called the graph of the equation.

An application of the implicit differentiation is to find the equation of the tangent line at (a,b) on the graph of an equation.

Example. We again consider $y^3+3x^2y=13$.

We have computed that $y' = \dfrac{-2xy}{x^2+y^2}$.

The point $(2,1)$ is on the graph of $y^3+3x^2y=13$

since $1^3+3\times2^2\times1=13$.

So the slope of the tangent line at $(2,1)$ is

$$y' = \frac{-2\times2\times1}{2^2+1^2} = -\frac{4}{5}$$

So the equation of the tangent line at $(2,1)$ is

$$y-1=-\frac{4}{5}(x-2)$$

Example. Find the slope of the tangent line to the curve $2xy-3y^2=9$ at $(6,1)$.

$$(2xy-3y^2)'=0$$

$$2y+2xy'-3\cdot2y\cdot y'=0$$

So $y' = \dfrac{y}{3y-x}$. when $\begin{cases}x=6\\y=1\end{cases}$ $y' = \dfrac{1}{3\times1-6} = -\dfrac{1}{3}$

Higher Order Implicit Differentiation:

Sometimes we also need to know about the higher order derivatives of a given implicit function. In this case, we need to differentiate the equation n times, each time obtain a higher order derivative.

Example. We again consider $y^3+3x^2y=13$.

We've obtained $3y^2y'+6xy+3x^2y'=0$, $y'=\dfrac{-2xy}{x^2+y^2}$.

In order to obtain y'', there are two ways:

Method I: differentiate both sides of $3y^2y'+6xy+3x^2y'=0$.

We get $(3y^2y'+6xy+3x^2y')'=0$

$$3\cdot(y^2)'y'+3y^2(y')'+6(x)'y+6xy'+3(x^2)'y'+3x^2(y')'=0$$

$$3\cdot 2y\cdot y'\cdot y'+3y^2y''+6y+6xy'+3\cdot 2x\cdot y'+3x^2y''=0$$

$$3(x^2+y^2)y''=-6\,[y\cdot(y')^2+y+2xy']$$

$$(x^2+y^2)y''=-2\cdot\left[y\cdot\frac{4x^2y^2}{(x^2+y^2)^2}+y+2x\cdot\frac{-2xy}{x^2+y^2}\right]$$

$$(x^2+y^2)y''=-\frac{2y}{(x^2+y^2)^2}\cdot\left[4x^2y^2+(x^2+y^2)^2-4x^2(x^2+y^2)\right]$$

$$y''=-\frac{2y}{(x^2+y^2)^3}\left[(x^2+y^2)^2-4x^4\right]$$

$$=-\frac{2y}{(x^2+y^2)^3}\cdot(3x^2+y^2)(y^2-x^2)$$

Method II : differentiate both sides of $y' = \frac{-2xy}{x^2+y^2}$.

$$y'' = \left(\frac{-2xy}{x^2+y^2}\right)'$$

$$= -2 \cdot \left(\frac{xy}{x^2+y^2}\right)'$$

$$= -2 \cdot \frac{(xy)'(x^2+y^2) - xy(x^2+y^2)'}{(x^2+y^2)^2}$$

$$= -2 \cdot \frac{(y+xy')(x^2+y^2) - xy(2x+2y\cdot y')}{(x^2+y^2)^2}$$

$$= -2 \cdot \frac{\left(y+x\cdot\frac{-2xy}{x^2+y^2}\right)(x^2+y^2) - 2xy\left(x+y\cdot\frac{-2xy}{x^2+y^2}\right)}{(x^2+y^2)^2}$$

$$= \frac{-2}{(x^2+y^2)^2}\cdot\left[y(x^2+y^2) - 2x^2y - 2x^2y + \frac{4x^2y^3}{x^2+y^2}\right]$$

$$= \frac{-2}{(x^2+y^2)^2}\left[y(x^2+y^2) - 4x^2y + \frac{4x^2y^3}{x^2+y^2}\right]$$

$$= \frac{-2}{(x^2+y^2)^2}\quad\frac{y(x^2+y^2)^2 - 4x^2y(x^2+y^2) + 4x^2y^3}{x^2+y^2}$$

$$= \frac{-2y}{(x^2+y^2)^3}\left[(x^2+y^2)^2 - 4x^2(x^2+y^2) + 4x^2y^2\right]$$

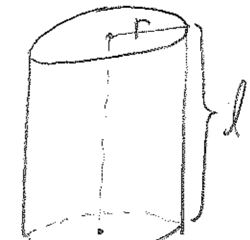
$$= \frac{-2y}{(x^2+y^2)^3}\left[x^4+2x^2y^2+y^4-4x^4-4x^2y^2+4x^2y^2\right]$$

$$= \frac{-2y}{(x^2+y^2)^3}\left[y^4+2x^2y^2-3x^4\right]$$

$$= \frac{-2y}{(x^2+y^2)^3}\cdot(3x^2+y^2)(y^2-x^2)$$

Example. The total area of the surface of a cylinder with radius r and height l is given by the formula

$$S = 2\pi r \cdot l + \pi r^2 \quad (*)$$



If we plan to make a cylinder with a fixed amount of material, then S is a fixed number. In this case, the height l determines the radius r.

so we can regard r as a function of l: $r = f(l)$

Then taking derivative on both sides of $(*)$ w.r.t. l:

$$0 = 2\pi(r' \cdot l + r) + \pi \cdot 2r \cdot r'$$

So $$r' = \frac{-r}{r+l}$$

The intuition is if we increase the height of a big cylinder by 1 unit while keeping the total area of surface fixed, we need to reduce the radius by about $\frac{r}{r+l}$ units

Example. The national Income of a closed economy is Y, which is the sum of consumption C and investment I.

i.e. $Y = C + I$

The consumption C is a function of national income

$C = f(Y)$

$f'(Y)$ is called the marginal propensity to consume

Then we see $Y = f(Y) + I$

Suppose this equation defines Y as a function of I.

Then by implicit differentiation:

$$Y' = f'(Y) \cdot Y' + 1$$

so $$Y' = \frac{1}{1 - f'(Y)}$$

Take derivative again on $Y' = f'(Y)Y' + 1$, we get

$$Y'' = f''(Y) \cdot Y' \cdot Y' + f'(Y) Y''$$

so $$Y'' = \frac{f''(Y)(Y')^2}{1 - f'(Y)} = \frac{f''(Y)}{(1 - f'(Y))^3}$$

Example. An object of mass m is moving vertically. It's kinetic energy is $E_K = \frac{1}{2} m v^2$, where v is the velocity, and its potential energy is $E_u = mgh$, where h is the altitude, and g is a constant. The total mechanical energy $E = E_K + E_u$ is conserved if no external force acting on the object except gravity. Then the velocity v is a function of altitude, given by $\frac{1}{2} m v^2 + mgh = E$

so. $\frac{1}{2} m \cdot 2v \cdot \frac{dv}{dh} + mg = 0$

$$\frac{dv}{dh} = -\frac{g}{v}$$

Inverse Differentiation

Recall that if a function f is one-to-one with domain A and B, then f has an inverse function g, whose domain is B and range is A, satisfying $y=f(x) \Leftrightarrow x=g(y)$

Theorem. If f is strictly increasing, then its inverse function is also strictly increasing; If f is strictly decreasing, the its inverse function is also strictly decreasing

Proof. If f is strictly increasing,

for any $y_1<y_2$, let $x_1=g(y_1)$, $x_2=g(y_2)$

then $y_1=f(x_1)$, $y_2=f(x_2)$

$y_1<y_2 \Rightarrow f(x_1)<f(x_2)$

and f is increasing implies $x_1<x_2$

so $g(y_1)<g(y_2)$. g is also increasing

Theorem. If f is differentiable at x_0, $f'(x_0)\neq 0$, and f has an inverse function g, $y_0=f(x_0)$, then g is differentiable at y_0,

and $g'(y_0)=\dfrac{1}{f'(x_0)}$

Proof. $g(f(x))=x$ so

$g'(f(x))\cdot f'(x)=1$

so $g'(f(x))=\dfrac{1}{f'(x)}$

Example. $f(x) = x^5 + 3x^3 + 6x - 3$. First show that f has an inverse function g, then compute $g'(-3)$.

$f'(x) = 5x^4 + 9x^2 + 6 > 0$ for all x.

So $f(x)$ is strictly increasing, hence has an inverse function g.

$g'(f(x)) = \frac{1}{f'(x)}$ Note that $f(0) = -3$

So $g'(f(0)) = \frac{1}{f'(0)}$

$g'(-3) = \frac{1}{6}$

We can also obtain high order differentiation of the inverse function g of a function f.

Example. We know $g'(f(x)) \cdot f'(x) = 1$.

If we differentiate the equation again, we get:

$$(g'(f(x)))' \cdot f'(x) + g'(f(x))(f'(x))' = 0$$

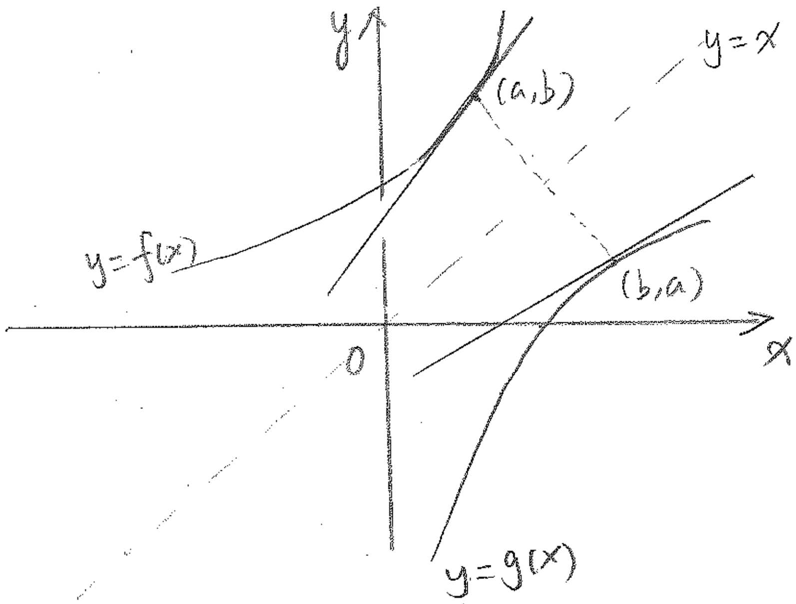
$$g''(f(x)) \cdot f'(x) \cdot f'(x) + g'(f(x)) \cdot f''(x) = 0$$

So
$$g''(f(x)) = \frac{-f''(x)}{(f'(x))^2} \cdot g'(f(x))$$

$$= \frac{-f''(x)}{(f'(x))^2} \cdot \frac{1}{f'(x)}$$

$$= -\frac{f''(x)}{(f'(x))^3}$$

Geometric Illustration of derivative of inverse function:



$b=f(a) \Leftrightarrow a=g(b)$

So (a,b) is on graph of f $\Leftrightarrow$ (b,a) is on graph of g

The graphs of f and g are symmetric with respect to the line $y=x$, so the tangent line of f at (a,b) and the tangent line of g at (b,a) have slopes that are inverse of each other. And the tangent lines are also symmetric with respect to $y=x$!

Example. $f(x) = x^7 + x^5 + x^3 + 2x + 1$

Find the equation of the tangent line of its inverse function g at $(1,0)$

Method I:

$$g'(f(x)) = \frac{1}{f'(x)}$$

$$= \frac{1}{7x^6 + 5x^4 + 3x^2 + 2}$$

So $g'(1) = g'(f(0)) = \frac{1}{f'(0)} = \frac{1}{2}$

So the tangent line is $y - 0 = \frac{1}{2}(x-1)$, i.e. $y = \frac{1}{2}(x-1)$

Method II: First find the equation of the tangent line of f at $x=0$

$f'(x) = 7x^6 + 5x^4 + 3x^2 + 2 \Rightarrow f'(0) = 2 \Rightarrow y - 1 = 2(x-0)$

Next switch x and y in the previous equation:

$2y = x-1 \Rightarrow y = \frac{1}{2}(x-1)$