1. Prove $\sum_{n=0}^{\infty} z^n$ does NOT converge uniformly on the domain $0 < |z| < 1$.

   Solution: When $\epsilon = 1$, for any $N \in \mathbb{N}$, consider the real polynomial $f(x) = x^{N+1} - (1-x)$. Since $f(1) = 1$, by the continuity of $f(x)$, $f(x) > 0$ in a neighbourhood of $x = 1$, so there exists some number $0 < x_N < 1$ such than $f(x_N) > 0$, i.e. $\frac{x_N^{N+1}}{1-x_N} > 1$.

   We thus see for any $N \in \mathbb{N}$, we can find $z = x_N$ in the domain $0 < |z| < 1$ such that

   $$|\frac{1}{1-x_N} - \sum_{n=0}^{N} x_N| = \frac{x_N^{N+1}}{1-x_N} > 1 = \epsilon$$

   So the series doesn't converge uniformly on this domain.

2. Show that the function defined by

   $$f(z) = \begin{cases} \frac{1-\cos z}{z^2}, (z \neq 0) \\ \frac{1}{2}, (z = 0) \end{cases}$$

   is entire.

   $\cos z = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} z^{2n}$ for all $z \in \mathbb{C}$, so

   $$f(z) = \frac{1-\cos z}{z^2} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n)!} z^{2n-2} = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n+2)!} z^{2n}$$

   for all $z \neq 0$. But observe when $z = 0$, the power series also equals to $f(0)$, we get

   $$f(z) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n+2)!} z^{2n}$$

   for all $\mathbb{C}$, so the radius of convergence is ∞, which implies $f(z)$ is analytic everywhere on $\mathbb{C}$.

3. Find the Taylor series expansion for $f(x) = \frac{1}{(1-z)^2}$ at $z_0 = 0$ by differentiating the Taylor series expansion $\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$ $(|z| < 1)$

Solution: We know within the circle of convergence we can differentiate the power series by differetiating term by term.

$$(\frac{1}{1-z})' = \sum_{n=0}^{\infty}(z^n)'$$

$$\frac{1}{(1-z)^2} = \sum_{n=1}^{\infty} nz^{n-1} = \sum_{n=0}^{\infty}(n+1)z^n$$

4. m is a positive integer. Prove that if f is analytic at z_0 and $f(z_0) = f'(z_0) = ... = f^{(m)}(z_0) = 0$, then the function

$$g(z) = \begin{cases} \frac{f(z)}{(z-z_0)^{m+1}}, (z \neq z_0) \\ \frac{f^{(m+1)}(z_0)}{(m+1)!}, (z = z_0) \end{cases}$$

is analytic at z_0.

Solution:

Since f is analytic at z_0, there is an open disk D centred at z_0 such that f is analytic on D, so for any $z \in D$

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}}{n!}(z-z_0)^n = \sum_{n=m+1}^{\infty} \frac{f^{(n)}}{n!}(z-z_0)^n$$

Then for any $z \in D - \{z_0\}$,

$$g(z) = \frac{f(z)}{(z-z_0)^{m+1}} = \sum_{n=m+1}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z-z_0)^{n-(m+1)} = \frac{f^{(m+1)}(z_0)}{(m+1)!} + \sum_{n=1}^{\infty} \frac{f^{(n+m+1)}(z_0)}{(n+m+1)!}(z-z_0)^n$$

and observe $g(z_0) = \frac{f^{(m+1)}(z_0)}{(m+1)!}$ also satisfies the series. We conclude for any $z \in D$,

$$g(z) = \frac{f^{(m+1)}(z_0)}{(m+1)!} + \sum_{n=1}^{\infty} \frac{f^{(n+m+1)}(z_0)}{(n+m+1)!}(z-z_0)^n$$

So D is within the circle of convergence of the series, which implies $g(z)$ is analytic on D.

5. Find the Laurent series expansion of

$$f(z) = \frac{1}{(z-1)(z-2)}$$

in the domain $1 < |z| < 2$

Solution:

$$\begin{aligned} f(z) =& \frac{1}{(z-1)(z-2)} \\ =& \frac{1}{1-z} - \frac{1}{2-z} \\ =& -\frac{1}{z} \times \frac{1}{1-\frac{1}{z}} - \frac{1}{2} \times \frac{1}{1-\frac{z}{2}} \\ =& -\frac{1}{z}\sum_{n=0}^{\infty}(\frac{1}{z})^n - \frac{1}{2}\sum_{n=0}^{\infty}(\frac{z}{2})^n \\ =& -\sum_{n=1}^{\infty}\frac{1}{z^n} - \sum_{n=0}^{\infty}\frac{1}{2^{n+1}}z^n \end{aligned}$$

6. $f(z) = \frac{1}{z-z^2}$.

(i). Find the Laurent series expansion of $f(z)$ in the domain $0 < |z| < 1$

Solution:

$$\begin{aligned} f(z) =& \frac{1}{z-z^2} \\ =& \frac{1}{z} \times \frac{1}{1-z} \\ =& \frac{1}{z}\sum_{n=0}^{\infty} z^n \\ =& \sum_{n=-1}^{\infty} z^n \end{aligned}$$

(ii). Find the Laurent series expansion of $f(z)$ in the domain $1 < |z| < \infty$

Solution:

$$\begin{aligned} f(z) =& \frac{1}{z - z^2} \\ =& -\frac{1}{z^2} \times \frac{1}{1 - \frac{1}{z}} \\ =& -\frac{1}{z^2} \sum_{n=0}^{\infty} \frac{1}{z^n} \\ =& -\sum_{n=2}^{\infty} \frac{1}{z^n} \end{aligned}$$

(iii). Find the Laurent series expansion of $f(z)$ in the domain $0 < |z - 1| < 1$

Solution:

$$\begin{aligned} f(z) =& \frac{1}{z - z^2} \\ =& -\frac{1}{z-1} \times \frac{1}{z} \\ =& -\frac{1}{z-1} \times \frac{1}{1 + (z-1)} \\ =& -\frac{1}{z-1} \sum_{n=0}^{\infty} (-1)^n (z-1)^n \\ =& \sum_{n=-1}^{\infty} (-1)^n (z-1)^n \end{aligned}$$

(iv). Find the Laurent series expansion of $f(z)$ in the domain $1 < |z - 1| < \infty$

Solution:

$$
\begin{aligned}
f(z) =& \frac{1}{z - z^2} \\
=& -\frac{1}{(z-1)^2} \times \frac{1}{1 + \frac{1}{z-1}} \\
=& -\frac{1}{(z-1)^2} \sum_{n=0}^{\infty} (-1)^n \frac{1}{(z-1)^n} \\
=& \sum_{n=2}^{\infty} (-1)^n \frac{1}{(z-1)^n}
\end{aligned}
$$